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ON THE UNIFORMLY MOST POWERFUL INVARIANT TEST FOR THE SHOULDER CONDITION IN LINE TRANSECT SAMPLING

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SUMMARY

In wildlife population studies one of the main goals is estimating the population density. Line transect sampling is a well established methodology for this purpose. The usual approach for estimating the density of the population of interest is to assume a particular model for the detection function. The estimates are extremely sensitive to the shape of the detection function, particularly to the so-called shoulder condition, which ensures that an animal is nearly certain to be detected if it is at a small distance from the observer. For instance, the half-normal model satisfies this condition whereas the negative exponential does not. So, testing whether the shoulder condition is consistent with the data is a primary concern. Since the problem of testing such a hypothesis is invariant under the group of scale transformations, in this paper we propose the uniformly most powerful test in the class of the scale invariant tests for the half-normal model against the negative exponential model. The asymptotic distribution of the test statistic is derived. The critical values and the power are tabulated via Monte Carlo simulations for small samples.

Keywords: *Line Transect Sampling, Shoulder Condition, Uniformly Most Powerful Invariant Test, Asymptotic Critical Values, Monte Carlo Critical Values.*

1. INTRODUCTION

Many studies of wildlife populations aim at estimating the population abundance. Transect sampling provides an effective approach for the estimation of the population size ν or the density $\delta = \nu / A$, where A is the area of the study region. A thorough review of this methodology is given by Buckland *et al.* (2001, chap. 1). The line transect design in particular assumes that:

- k not overlapping lines are randomly chosen within the study area;
- animals of interest are uniformly distributed with respect to perpendicular distance from the lines;
- along each of the selected lines, an observer measures the distance from the line to any animal detected;
- animals on the lines are detected with certainty;

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- animals are detected at their initial location, prior to any movement;
- distances are measured without errors;
- detections are independent events.

Since the number of animals observed from each line is quite small in many contexts where this sample scheme is adopted, sampled distances are pooled together to increase the sample size.

Let $z_1, \dots, z_n$ be the sample of size n obtained by pooling together the distances from each of the k lines. Let f be the probability density function (pdf) of the observed distances and let g be the detection function, that is to say $g(y)$ is the conditional probability of detecting an animal given that it is at distance y from the line. Given the above assumptions it turns out that the relation

$$f(z) = \frac{g(z)}{\int_0^{+\infty} g(y)dy} \tag{1}$$

holds for every distance z and the general estimator of the population density δ is

$$\hat{\delta} = \frac{n}{2l}\hat{f}(0),$$

where l is the sum of the length of the considered lines and $\hat{f}(0)$ is an estimator of f at 0 which satisfies the fundamental identity

$$f(0) = \frac{1}{\int_0^{+\infty} g(y)dy}.$$

The basic problem for estimating δ, or equivalently ν, is therefore to estimate $f(0)$. A thorough discussion of assumptions and formulas reported above can be found in Buckland *et al.* (2001, chap. 2). We consider two popular families of detection functions (Zhang, 2001; Eidous, 2005): the half-normal family

$$g(y) = \exp\left(-\frac{y^2}{2\sigma^2}\right) \quad (\sigma > 0) \tag{2}$$

and the negative exponential family

$$g(y) = \exp\left(-\frac{y}{\sigma}\right) \quad (\sigma > 0). \tag{3}$$

The former satisfies the shape criterion

$$g'(0) = 0 \tag{4}$$

whereas the latter does not. This property, also known as the shoulder condition, ensures that animal detection is nearly certain at small distances from the observer (Buckland *et al.*, 2001, 2004). However, such a condition fails when detectability decreases sharply around the observation lines because of low or inexistent visibility (e.g. in presence of fog or dense vegetation) and it cannot hold for transect data of many wildlife species (Mack and Quang, 1998; Mack *et al.*, 1999). In the line transect framework, Eidous (2005) reported some simulation results suggesting that the usual estimators of δ are extremely sensitive to departures from the shape criterion (4). Hence evaluating whether the shape criterion is consistent with the data should be a preliminary step for any attempt to estimate wildlife population density via line transect sampling (Zhang, 2003; Eidous, 2005). This problem has been previously addressed by Mack (1998) and Zhang (2001) although these authors did not discussed the optimal properties of their procedures.

In this paper we propose an optimal procedure for testing the shoulder condition (4). As this condition is independent from the choice of the measure unit for the distance, the scale invariance seems to be quite a natural restriction for a statistical test. Particularly we consider a scale invariant test for discriminating between the two families (2) and (3). Because of (1) this turns out to be equivalent to testing that the distance pdf belongs to one of the two families

$$F_0 = \left\{ f(z) = \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{z^2}{2\sigma^2} \right) : \sigma > 0 \right\} \tag{5}$$

and

$$F_1 = \left\{ f(z) = \frac{1}{\sigma} \exp\left(-\frac{z}{\sigma} \right) : \sigma > 0 \right\}. \tag{6}$$

The test proposed herein is the uniformly most powerful (UMP) in the class of the scale invariant tests. This is discussed in the following section where the asymptotic distributions of the test statistic under (5) and (6) are calculated. In section 3 the critical values and the powers of the test are tabulated by Monte Carlo simulations for several typical α-levels and small sample sizes n. Conclusions are provided in section 4.

2. THE UMP SCALE INVARIANT TEST

Given n independent observations $z_1,\dots, z_n$ from a random variable Z with an unknown pdf f, we consider the problem of testing

$$H_0 : f \in F_0 \qquad \text{vs.} \qquad H_1 : f \in F_1, \tag{7}$$

where F_0 is the family of half-normal distributions with scale parameter σ, and F_1 is the family of Gamma distributions with shape parameter 1 and scale parameter σ, specified in (5) and (6) respectively. This problem is invariant under the group of scale transformations

$$G = \{\gamma(z) = rz : r > 0\}.$$

A maximal invariant under G (Lehmann and Romano, 2005, p. 213) is

$$(z_1/z_n, z_2/z_n, \dots, z_{n-1}/z_n) \tag{8}$$

It can be proved (see Appendix A) that the UMP test among all the invariant functions, i.e. the functions of this maximal invariant (Lehmann and Romano, 2005, p. 214) rejects the null hypothesis for large values of the likelihood ratio

$$\lambda = \frac{(n-1)(\pi/n)^{n/2}}{2^{n-1}\Gamma(n/2)} \left[\frac{\frac{1}{n}\sum_{i=1}^{n} z_i^2}{\left(\frac{1}{n}\sum_{i=1}^{n} z_i\right)^2} \right]^{n/2}. \tag{9}$$

Given that random variable corresponding to λ is a monotonically increasing function of the statistic

$$Q_n = \frac{\frac{1}{n}\sum_{i=1}^{n} Z_i^2}{\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)^2}, \tag{10}$$

the critical region of the UMP scale invariant test for the hypotheses (7) is

$$Q_n \geq q_{n,\alpha}, \tag{11}$$

where α denotes the level of significance and $q_{n,\alpha}$ is the corresponding critical value so that

$$P\left(Q_n \geq q_{n,\alpha} | H_0\right) = \alpha.$$

It may be observed that the test procedure is equivalent to the likelihood ratio test proposed by Zhang (2001) although the UMP invariant property was not considered in that paper.

Furthermore, the asymptotic normal distribution under H_0 is

$$\sqrt{n}(Q_n - \pi/2) \xrightarrow{d} N\left(0, \pi^2(\pi-3)/2\right), \tag{12}$$

and under H_1

$$\sqrt{n}(Q_n - 2) \xrightarrow{d} N(0,4), \tag{13}$$

which are derived from the bivariate central limit theorem and the delta method (see the Appendix B). For large n the approximate critical value and the power are given respectively by

$$q_{n,\alpha} \cong 1.57 + 0.84 z_{1-\alpha}/\sqrt{n}$$

and

$$1-\beta = P\left(Q_n \geq q_{n,\alpha} | H_1\right) \cong 1 - \Phi\left(0.42 z_{1-\alpha} - 0.21\sqrt{n}\right),$$

where $z_{1-\alpha}$ is the $(1-\alpha)$-th quantile and Φ is the cumulative distribution function of the standard normal distribution. Hence, the proposed test is also consistent (Lehmann, 2001, p. 158).

3. TABLES OF CRITICAL VALUES AND POWERS

In this section Monte Carlo simulations are performed in order to obtain the empirical critical values and powers for small sample sizes.

The simulation design consists of randomly drawing n distances from the distribution (5) setting $\sigma = 1$. We can make this without loss of generality as the distribution of the test statistic under (5) or (6) does not depend on the scale parameter.

The statistic (10) is then applied to each of the simulated samples and the procedure is repeated 5000 times. The critical value $q_{n,\alpha}$ for a considered significance level α is obtained as $100 \times (1 - \alpha)$-th percentile of the Monte Carlo replicates. We obtained the power of the test analogously by simulating each sample according to the alternative distribution (6). Monte Carlo approximations of the critical values $q_{n,\alpha}$ and powers are reported in Table 1 and in Table 2. The power obtained under (6) is good even in the case of a small sample and low α.

TABLE 1. - *Empirical critical values $q_{n,\alpha}$ of the UMP scale invariant test*

α	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n = 100$
0.01	1.998	1.946	1.879	1.840	1.788
0.05	1.815	1.793	1.768	1.754	1.708
0.10	1.748	1.734	1.709	1.703	1.674

TABLE 2. - *Empirical powers $1-\beta$ of the UMP scale invariant test*

α	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n = 100$
0.01	0.336	0.439	0.581	0.667	0.859
0.05	0.601	0.682	0.774	0.815	0.953
0.10	0.706	0.776	0.859	0.887	0.976

Table 3 and Table 4 show the asymptotical critical values and powers for the same test. It turned out that the simulated critical values and powers are very similar to those obtained by the asymptotic distribution for a sample size as big as 100.

TABLE 3. - *Asymptotical critical values $q_{n,\alpha}$ of the UMP scale invariant test*

α	$n = 30$	$n = 40$	$n = 50$	$n = 60$	$n = 100$
0.01	1.926	1.878	1.846	1.822	1.765
0.05	1.822	1.788	1.765	1.748	1.708
0.10	1.766	1.740	1.722	1.709	1.678

TABLE 4. - *Asymptotical powers* $1-\beta$ *of the UMP scale invariant test*

α	$n=30$	$n=40$	$n=50$	$n=60$	$n=100$
0.01	0.580	0.650	0.707	0.755	0.880
0.05	0.687	0.749	0.797	0.835	0.928
0.10	0.739	0.794	0.837	0.870	0.946

The test performs reasonably well in terms of the power in the case of the data generated from a mixture of (6) and (5) too. In particular we consider the case where the sample is drawn from the following pdf

$$p\ exp(-z) + (1-p)\ \sqrt{\frac{2}{\pi}}\ exp\left(-z^2/2\right),$$

where p is the average proportion of the observed distances simulated from a population distributed according to the alternative hypothesis in (7), whereas $1-p$ is the average proportion of distances drawn from the null hypothesis in (7). In order to illustrate the behaviour of the test, Table 5 shows the power at level $\alpha = 0.05$ for a range of mixture proportions p and some sample sizes. It is not surprising that the power in Table 5 is quite smaller than the corresponding power in Table 2 particularly for small sample size.

It can be observed that the power of the test increases as p increases. This is appreciable since the probability of correctly rejecting the null hypothesis gets bigger as the proportion of the data drawn under the alternative hypothesis rises.

TABLE 5. - *Empirical powers* $1-\beta$ *of the UMP scale invariant test for different mixture proportions and for* $\alpha = 0.05$

p	$n=40$	$n=50$	$n=100$
0.25	0.261	0.299	0.454
0.50	0.442	0.517	0.740
0.75	0.580	0.656	0.887

4. CONCLUSIONS

In line transect sampling the problem of testing the shoulder condition of a detection function is invariant under the group of scale transformations. Hence, the scale invariance is a natural restriction on the statistical procedure one has to use. In the case of the half-normal and the negative exponential family, two commonly used models of detection functions, the above problem is reduced to testing (7). In this paper we proposed the UMP scale invariant test for the abovementioned problem and the limiting normal distribution of the test statistic is provided. For small samples we tabulated the critical values and related powers via Monte Carlo simulations for a range of different sample sizes and significant levels. It turned out that the simulated critical values and powers are very similar to those obtained by the asymptotic distribution for a sample size of 100 or more. Finally, it can be observed that the proposed procedure performs well also in the case of a sample drawn from a mixture of the negative exponential distribution and the half-normal distribution.

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RIASSUNTO

L'obiettivo di molti studi ecologici consiste nella stima della densità di una popolazione animale. A questo proposito, il campionamento tramite transetto lineare costituisce una metodologia molto diffusa. Per stimare la densità di una popolazione biologica si assume solitamente un particolare modello per la funzione di avvistamento. D'altronde, le stime risultano fortemente influenzate dalla forma della funzione di avvistamento e, in particolare, dalla cosiddetta "shoulder condition", che assicura un'alta probabilità di avvistamento degli animali che si trovano a breve distanza dall'osservatore. Assume quindi particolare rilievo verificare se questa condizione è supportata dai dati. Due esempi di famiglie di funzioni di avvistamento utilizzate in letteratura sono rappresentate dal modello seminormale e da quello esponenziale negativo. Poiché il primo modello, a differenza del secondo, soddisfa alla "shoulder condition", che risulta invariante rispetto ai cambiamenti di scala, in questo lavoro si fornisce il test uniformemente più potente fra gli invarianti per cambiamento di scala per saggiare i due suddetti modelli. Inoltre, si determina la distribuzione asintotica e, nel caso di piccoli campioni, si calcolano i valori critici e la potenza del test con il metodo Monte Carlo..

APPENDIX A

The pdf of the sample $(z_1, ..., z_n)$ can be written as:

$$L(z_1, ..., z_n) = \frac{1}{\sigma^n} \prod_{i=1}^{n} f\left(\frac{z_i}{\sigma}\right),$$

where:

$$f(z) = \begin{cases} \sqrt{\frac{2}{\pi}} \exp(-z^2/2) & : \text{under } H_0 \\ \exp(-z) & : \text{under } H_1 \end{cases}.$$

Hence the maximal invariant (8) is expressed as:

$$(r_1, ..., r_{n-1}) = (z_1/z_n, z_2/z_n, ..., z_{n-1}/z_n)$$

and has pdf given by:

$$\int_0^{+\infty} z_n^{n-1} L(z_n r_1, ..., z_n r_{n-1}, z_n) dz_n = z_n^n \int_0^{+\infty} u^{n-1} \prod_{i=1}^{n} f(z_i u) du$$

$$= \begin{cases} \dfrac{2^{n-1}\Gamma(n/2)}{\left(\pi \sum_{i=1}^{n} z_i^2\right)^{n/2}} & : \text{under } H_0 \\ \dfrac{(n-1)!}{\left(\sum_{i=1}^{n} z_i\right)^{n}} & : \text{under } H_1 \end{cases}$$

from which the likelihood ratio (9) follows.

By the Neyman-Pearson Lemma the most powerful test rejects the null hypothesis when (9) is too large. Given that its critical region does not depend on σ, the test is UMP among all invariant tests.

APPENDIX B

By bivariate central limit theorem (Lehmann, 2001, p. 291),

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i - \mu_1, \frac{1}{n}\sum_{i=1}^{n} Z_i^2 - \mu_2\right) \xrightarrow{d} N(0,\ \Sigma),$$

where

$$\mu_1 = E(Z), \quad \mu_2 = E\left(Z^2\right), \quad 0 = (0,0)$$

and

$$\sum = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

with $\sigma_1^2 = Var(Z)$, $\sigma_2^2 = Var\left(Z^2\right)$ and $\sigma_{12} = Cov\left(Z, Z^2\right)$.

Then, by delta method (Lehmann, 2001, p. 296),

$$\sqrt{n}(Q_n - \eta) \xrightarrow{d} N\left(0, \tau^2\right),$$

where

$$\eta = \frac{\mu_2}{\mu_1^2}$$

and

$$\tau^2 = \frac{4\mu_2^2}{\mu_1^6}\sigma_1^2 - \frac{4\mu_2}{\mu_1^5}\sigma_{12} + \frac{1}{\mu_1^4}\sigma_2^2 = \frac{\mu_4 - \mu_2^2}{\mu_1^4} + 4\mu_2\frac{\mu_2^2 - \mu_1\mu_3}{\mu_1^6},$$

with $\mu_3 = E\left(Z^3\right)$ and $\mu_4 = E\left(Z^4\right)$.

Therefore, under H_0 we have

$$\nu = \pi/2 \quad \text{and} \quad \tau^2 = \pi^2(\pi - 3)/2,$$

whereas under H_1

$$\nu = 2 \quad \text{and} \quad \tau^2 = 4.$$

Hence, (12) and (13) follow.

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