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MIXTURE OF POLISICCHIO'S TRUNCATED PARETO DISTRIBUTIONS WITH BETA WEIGHTS

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SUMMARY

A new three-parameter density function $f(x:\mu;\alpha;\theta)$ for non-negative variables, obtained as a mixture of ''Polisicchio's truncated Pareto distributions'', is proposed. The expectation of $f(x:\mu;\alpha;\theta)$ is equal to the parameter $\mu > 0$. The new density has positive asymmetry and Paretian right tail. The variance is equal to $\frac{\mu^2}{3}\frac{\theta(\theta+1)}{[\alpha^2+\alpha(\theta-1)-\theta]}$. The moments, the upper and lower truncated moments (with truncation at $x=\mu$), are compact expressions of beta functions.

Keywords: *Income Distribution, Positive Asymmetry, Paretian Tail, Mixture Density, Beta Weights.*

1. INTRODUCTION

The themes of income distribution by size and income inequality measures were introduced by Pareto (1895, 1896, 1897), who proposed three models for income distribution. Pareto's work has been developed by many Italian economists and statisticians. In particular Benini (1897, 1901, 1905, 1906) remarked that the actual distributions of many economic phenomena, when graphed on the Pareto double logarithmic scale, had a better fit with a parabolic curve than with the Pareto linear (type I) model. The support of Pareto's and Benini's models is the interval $0 < x_0 \leq x$, in which x_0 represents the minimum income. Vinci (1921) suggested the use of the Pearson type V distribution to represent the income on interval $0 < x$.

Some years later Amoroso (1925) provided an income model characterized by four parameters. The Amoroso distribution, which was rediscovered by Stacy (1962), is known nowadays as the ''generalized Gamma distribution''.

Very interesting are the contributions of D'Addario. The first contribution (D'Addario, 1949) is a generating system of income distributions based on the ''transformations'' of some well known random variables (Normal, Gamma, etc.). He then proposed (D'Addario, 1939b) a second method to obtain income distributions based on the function $\Psi(x) = E(X > x)$, where $E(X > x)$ is the mean value of the incomes greater than x.

D'Addario's third important contribution is the so called ''invariants method'' for the estimation of the parameters of Pareto's models (D'Addario, 1934, 1939a). Ac-

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cording to this method the estimators of the parameters of the model are obtained by replacing some model characteristics – which are functions of the parameters of the model – by sample characteristics. D'Addario proposed using those characteristics that usually were employed in the description of income distributions: arithmetic mean, Gini concentration ratio, minimum income, etc.

Dagum (1977, 1980, 1983) specified a generating system of income distributions based on the behaviour of the elasticity of the cumulative distribution function. Many income models can be generated by means of Dagum's system. In particular Dagum derived from this system three models.

Dagum (1990) later specified some properties that income distributions must fulfil. The essential properties proposed by Dagum are:

i) the existence of only a small number of finite moments;
ii) the economic significance of the parameters;
iii) model flexibility to fit both uni-modal and zero-modal distributions;
iv) a good fit of the whole range of income;

Stoppa (1990) proposed a generalization of the classic Pareto distribution by introducing a power transformation of the Pareto cumulative distribution function. In a later paper, Stoppa (1993) specified a new four parameter model.

Finally, Colombi (1990) proposed a three parameter model, assuming that the income is given by the product of two independent components having respectively a Paretian and a log-Normal distribution.

A more detailed exposition of income distributions can be found in Kleiber and Kotz (2003), in Dagum and Zenga (1990), in Zenga (1987) and in Guerrieri, Guarini and Bonadies (1987).

In this paper, a new three parameter density function $f(x;\mu;\alpha;\theta)$ for non-negative variables, has been proposed. The expectation of the new density is equal to the parameter $\mu > 0$. The cumulative probability $F(x)$ in correspondence of $x = \mu$, is function of α and θ: $F(\mu) = h(\alpha;\theta)$. The mean deviation is equal to $2\mu[2F(\mu) - 1]$. The variance is given by $\frac{\mu^2}{3}\frac{\theta(\theta+1)}{\alpha^2+\alpha(\theta-1)-\theta}$. Moreover, the new density has positive asymmetry and a Paretian right tail. The new density $f(x;\mu;\alpha;\theta)$ is obtained as a mixture of particular truncated Pareto distributions (Polisicchio, 2008) with beta weights.

The paper is organized as follows. In Section 2 the new density is derived and it is shown that the new density is closed with respect to scale transformations. Section 3 examines the density for $\theta = 2; 3; 4$. Section 4 considers an example of density with θ non-integer. In Section 5 the moments of the new density are derived. Section 6 deals with the cumulative distribution function $F(x)$, for $x \leq \mu$. Section 7 deals with the lower and upper truncated moments with truncation at $x = \mu$. The mean deviation is obtained in section 8. Section 9 deals with Zenga's point inequality measure $A(\mu)$. In Section 10 by utilizing some equalities regarding $F(\mu)$ and the integral $\int_0^\mu xf(x:\mu;\alpha;\theta)dx$, a relationship between Beta functions is obtained. Finally, Section 11 is devoted to conclusions.

2. THE DENSITY $f(x:\mu;\alpha;\theta)$ OF THE MIXTURE

The density $f(x:\mu;\alpha;\theta)$ is obtained as mixture of the following particular truncated Pareto distributions recently introduced by Polisicchio (2008),

$$f(x:\mu;k) = \begin{cases} \dfrac{\sqrt{\mu}}{2}k^{0.5}(1-k)^{-1}x^{-(1.5)}, & \left(\mu k \le x \le \dfrac{\mu}{k}\right) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

$$(\mu > 0; 0 < k < 1)$$

with weights given by the Beta density

$$g(k:\alpha;\theta) = \begin{cases} \dfrac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}, & (\alpha > 0;\theta > 0), (0 < k < 1) \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In (2), $B(\alpha;\theta)$ denotes the Beta function.

Polisicchio (2008) has shown that only for the density (1) the Zenga's (2007) point inequality measure

$$A(x) = 1 - \frac{\overset{-}{\mu}(x)}{\overset{+}{\mu}(x)} \tag{3}$$

is constantly equal to $(1-k)$.

In (3) $\overset{-}{\mu}(x)$ and $\overset{+}{\mu}(x)$ are respectively the lower and upper mean of a non-negative continuous r.v. X on the support $[a;b:0<a<b]$ with distribution function $F(x)$ and density $f(x)$:

$$\begin{cases} \overset{-}{\mu}(x) = \dfrac{1}{F(x)}\displaystyle\int_a^x tf(t)dt, \\ \overset{+}{\mu}(x) = \dfrac{1}{(1-F(x))}\displaystyle\int_x^b tf(t)dt. \end{cases} \quad (a < x < b) \tag{4}$$

The expectation of density (1) is equal to μ. Note that for a fixed value of μ the support of the density (1) is a function of k. Figure 1 reports the supports of (1) for $\mu = 2$ and $0 < k < 1$.

Now it is necessary to point out that a single Polisicchio's r.v. is not able to represent "real" income distribution, essentially because:
i) its support is not the interval $(0;\infty)$ but one sub-set of it;
ii) the behaviour of its density function is decreasing;
iii) the characteristic to have Zenga's (2007) point inequality measure $A(x)$ constantly equal to $(1-k)$ is not realistic.

Anyway, note that for a fixed μ, the support of Polisicchio's r.v. tends to the interval $(0;\infty)$ for $k \to 0$. So, it is natural to try to get a new density function with a

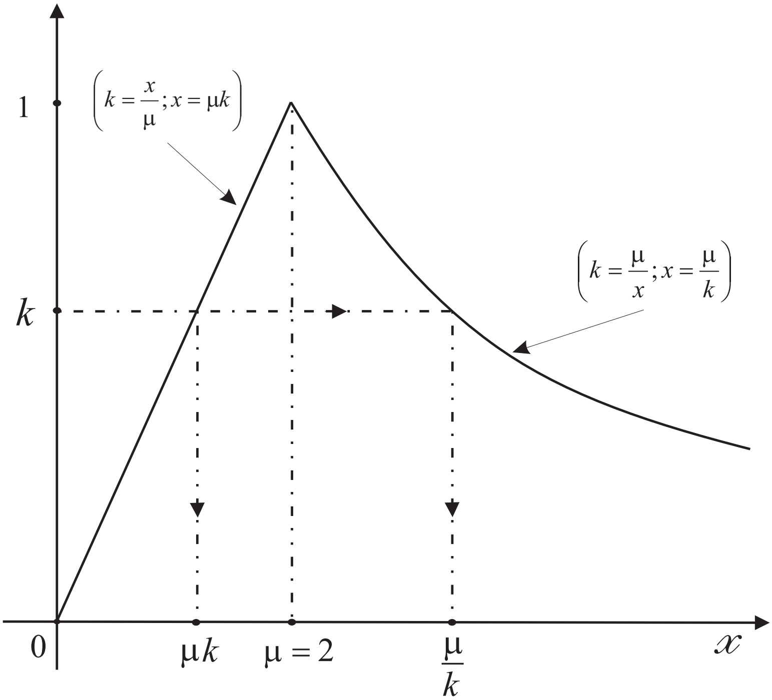

FIGURE 1. - *Supports of the density* $f(x;\mu;k)$

mixture of Polisicchio's r.v. with a fixed $\mu > 0$ and all the values of k in the interval $(0;1)$. Thus the resulting mixture r.v. has (finite) expectation equal to μ and support for the density the interval $(0;\infty)$. Obviously, the behaviour of the mixture density function is conditioned by the density of each Polisicchio's r.v. as well as by the density weights on the parameter k. For the density weights it has been chosen the Beta density $g(k:\alpha;\theta)$ because the shapes of $g(k:\alpha;\theta)$ are very broad: the shapes of the mixture density are expected to be broad too.

The density function $f(x:\mu;\alpha;\theta)$ of the mixture of Polisicchio's density $f(x:\mu;k)$, with weights on k furnished by the density $g(k:\alpha;\theta)$, is obtained by

$$f(x:\mu;\alpha;\theta) = \int_0^1 g(k:\alpha;\theta) f(x:\mu;k)dk, \quad (0 < x)$$

From the relation (1) it derives that:

i) for $\left\{(x < \mu k) \leftrightarrow \left(k > \frac{x}{\mu}\right)\right\} \rightarrow f(x:\mu;k) = 0$

ii) for $\left\{\left(x > \frac{\mu}{k}\right) \leftrightarrow \left(k > \frac{\mu}{x}\right)\right\} \rightarrow f(x:\mu;k) = 0.$

Consequently, $f(x:\mu;\alpha;\theta)$ is given by

$$f(x:\mu;\alpha;\theta)=\begin{cases}\displaystyle\int_0^{\frac{x}{\mu}} g(k:\alpha;\theta)f(x:\mu;k)dk, & (0<x<\mu)\\ \displaystyle\int_0^{\frac{\mu}{x}} g(k:\alpha;\theta)f(x:\mu;k)dk, & (\mu<x).\end{cases} \tag{5}$$

We will show now the function (5) is such that $\int_0^\infty f(x:\mu;\alpha;\theta)dx=1$.

From the properties of the density functions it follows that:

$$\int_0^1 g(k:\alpha;\theta)dk=1;$$

$$\int_{\mu k}^{\frac{\mu}{k}} f(x:\mu;k)dx=1.$$

Consequently

$$\int_0^1 g(k:\alpha;\theta)dk\int_{\mu k}^{\frac{\mu}{k}} f(x:\mu;k)dx=1$$

$$\int_0^1\left\{g(k:\alpha;\theta)\int_{\mu k}^{\frac{\mu}{k}} f(x:\mu;k)dx\right\}dk=1$$

$$\int_0^1\left\{\int_{\mu k}^{\frac{\mu}{k}} g(k:\alpha;\theta)f(x:\mu;k)dx\right\}dk=1$$

$$\int_0^1\left\{\int_{\mu k}^{\mu} g(k:\alpha;\theta)f(x:\mu;k)dx\right\}dk+\int_0^1\left\{\int_{\mu}^{\frac{\mu}{k}} g(k:\alpha;\theta)f(x:\mu;k)dx\right\}dk=1.$$

Changing the order of integration and utilizing the graph of Figure 1, it follows that

$$\int_0^{\mu}\left\{\int_0^{\frac{x}{\mu}} g(k:\alpha;\theta)f(x:\mu;k)dk\right\}dx+\int_{\mu}^{\infty}\left\{\int_0^{\frac{\mu}{x}} g(k:\alpha;\theta)f(x:\mu;k)dk\right\}dx=1. \tag{6}$$

End of the proof that the function (5) is a density on the support $(0;\infty)$.

Now, recalling the expressions of the densities $g(k:\alpha;\theta)$ and $f(x:\mu;\alpha;\theta)$, it derives that:

$$\int_0^{\mu}\left\{\frac{\sqrt{\mu}}{2B(\alpha;\theta)}x^{-1.5}\int_0^{\frac{x}{\mu}} k^{\alpha-1}(1-k)^{\theta-1}k^{0.5}(1-k)^{-1}dk\right\}dx+$$

$$+\int_{\mu}^{\infty}\left\{\frac{\sqrt{\mu}}{2B(\alpha;\theta)}x^{-1.5}\int_0^{\frac{\mu}{x}} k^{\alpha-1}(1-k)^{\theta-1}k^{0.5}(1-k)^{-1}dk\right\}dx=1$$

$$\Downarrow$$

$$\int_0^{\mu}\left\{\frac{\sqrt{\mu}}{2B(\alpha;\theta)}x^{-1.5}\int_0^{\frac{x}{\mu}}k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right\}dx+$$
$$+\int_{\mu}^{\infty}\left\{\frac{\sqrt{\mu}}{2B(\alpha;\theta)}x^{-1.5}\int_0^{\frac{\mu}{x}}k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right\}dx=1. \tag{7}$$

Consequently, the density of the mixture of the densities $f(x:\mu;k)$ with weights $g(k:\alpha;\theta)$ is given by

$$f(x:\mu;\alpha;\theta)=\begin{cases}\dfrac{1}{2\mu B(\alpha;\theta)}\left(\dfrac{x}{\mu}\right)^{-1.5}\displaystyle\int_0^{\frac{x}{\mu}}k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & (0<x<\mu)\\ \dfrac{1}{2\mu B(\alpha;\theta)}\left(\dfrac{\mu}{x}\right)^{1.5}\displaystyle\int_0^{\frac{\mu}{x}}k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & (\mu<x).\end{cases} \tag{8}$$

For $x=\mu$ the value of the density $f(x:\mu;\alpha;\theta)$ is given by

$$f(\mu:\mu;\alpha;\theta)=\begin{cases}\dfrac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)} & \text{for } \theta>1\ ,\\ \infty & \text{for } 0<\theta\le 1\ .\end{cases}$$

Consequently at $x=\mu$ the density $f(x:\mu;\alpha;\theta)$ is continuous for $\theta>1$. Now, denoting with

$$B(z:a;b)=\frac{1}{B(a;b)}\int_0^z k^{a-1}(1-k)^{b-1}dk,\quad (a>0,b>0)\ , \tag{9}$$

the incomplete beta function, the density (8) can be written as

$$f(x:\mu;\alpha;\theta)=\begin{cases}\dfrac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)}\left(\dfrac{x}{\mu}\right)^{-1.5}B\left(\dfrac{x}{\mu}:\alpha+0.5;\theta-1\right), & (0<x\le\mu)\\ \dfrac{B(\alpha+0.5;\theta-1)}{2\mu B(\alpha;\theta)}\left(\dfrac{\mu}{x}\right)^{1.5}B\left(\dfrac{\mu}{x}:\alpha+0.5;\theta-1\right), & (\mu<x).\end{cases} \tag{10}$$

Obviously, the use of (10) is allowed for:

$$\begin{array}{llll}\alpha>0 & \text{and } (\alpha+0.5)>0 & \to & \alpha>0\\ \theta>0 & \text{and } (\theta-1)>0 & \to & \theta>1.\end{array} \tag{11}$$

It is now useful to prove the following.

Lemma 1 *Let X be a random variable with density $f_X(x:1;\alpha;\theta)$, where*

$f_X(x:1;\alpha;\theta)$ is the density (8) for $\mu = 1$. Let $Y = \mu X$, $\mu > 0$. Then Y has density $f_Y(y:\mu;\alpha;\theta)$, where $f_Y(y:\mu;\alpha;\theta)$ is the density (8).

PROOF:
The proof is obtained easily remembering that, in the case of the transformation $Y = \mu X$, the density $f_Y(y)$ of Y is furnished by $f_Y(y) = \frac{1}{\mu} f_X\left(\frac{y}{\mu}\right)$, where $f_X(x)$ is the density of X.

3. DENSITY $f(x:\mu;\alpha;\theta)$ FOR $\theta = 2;3;4$

In this Section, to have an idea of the behavior of the mixture density, many graphs of $f(x:\mu;\alpha;\theta)$ are reported.

The evaluation of the integral

$$\int_0^z k^{\alpha+0.5-1}(1-k)^{\theta-2}dk$$

is very simple if θ is an integer ≥ 2. In effect, in this case,

$$(1-k)^{\theta-2} = [(-1)\cdot k + 1]^{\theta-2} = \sum_{j=0}^{\theta-2} \binom{\theta-2}{j}(-1)^j k^j \ .$$

Then, for θ integer ≥ 2, it derives that

$$\begin{aligned}\int_0^z k^{\alpha+0.5-1}(1-k)^{\theta-2}dk &= \int_0^z k^{\alpha+0.5-1}\sum_{j=0}^{\theta-2}\binom{\theta-2}{j}(-1)^j k^j dk \\ &= \sum_{j=0}^{\theta-2}\binom{\theta-2}{j}(-1)^j \int_0^z k^{\alpha+0.5-1+j}dk \\ &= \sum_{j=0}^{\theta-2}\binom{\theta-2}{j}(-1)^j \frac{z^{(\alpha+0.5+j)}}{(\alpha+0.5+j)} \ . \end{aligned} \tag{12}$$

For:

- $\theta = 2$ the value of (12) is:

$$\int_0^z k^{\alpha+0.5-1}dk = \frac{z^{\alpha+0.5}}{(\alpha+0.5)}; \tag{13}$$

- $\theta = 3$ the value of (12) is:

$$\begin{aligned}\int_0^z k^{\alpha+0.5-1}(1-k)dk &= \int_0^z k^{\alpha+0.5-1}dk - \int_0^z k^{\alpha+0.5}dk \\ &= \left\{\frac{z^{\alpha+0.5}}{(\alpha+0.5)} - \frac{z^{\alpha+1.5}}{\alpha+1.5}\right\}; \end{aligned} \tag{14}$$

- $\theta = 4$ the value of (12) is:

$$\int_0^z k^{\alpha+0.5-1}(1-k)^2 dk = \int_0^z k^{\alpha+0.5-1}(1-2k+k^2)dk$$
$$= \int_0^z k^{\alpha+0.5-1} - 2k^{\alpha+0.5} + k^{\alpha+1.5} dk$$
$$= \left\{ \frac{z^{\alpha+0.5}}{(\alpha+0.5)} - 2\frac{z^{\alpha+1.5}}{\alpha+1.5} + \frac{z^{\alpha+2.5}}{\alpha+2.5} \right\}. \tag{15}$$

3.1 *Density* $f(x : \mu; \alpha; 2)$

$$f(x : \mu; \alpha; 2) = \begin{cases} \frac{1}{2}\frac{(\alpha+1)}{(\alpha+0.5)}\left\{\alpha\frac{x^{\alpha-1}}{\mu^{\alpha}}\right\} & , \quad (0 < x \leq \mu) \\ \frac{1}{2}\frac{(\alpha)}{(\alpha+0.5)}\left\{(\alpha+1)\mu^{\alpha+1}x^{-[(\alpha+1)+1]}\right\}, & (\mu < x). \end{cases} \tag{16}$$

Note that for $(0 < x \leq \mu)$ the expression in curly brackets is a power density, while for $(\mu < x)$ the expression in curly brackets is a Pareto density with parameters μ and $(\alpha + 1)$.

In Figure 2 the graphs of the density (16) for $\mu = 2$ and $\alpha = 0.5; 1; 2; 3; 4$ are reported.

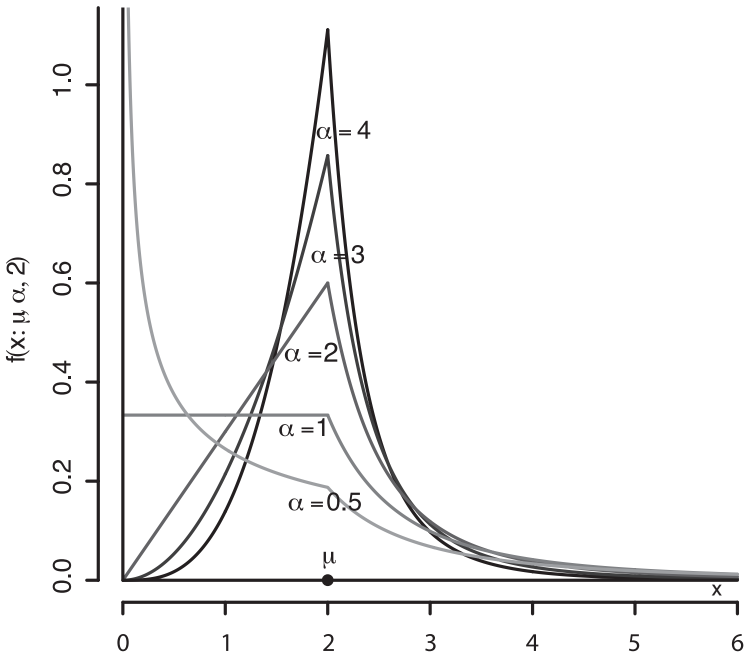

FIGURE 2. - *Graphs of* $f(x : \mu; \alpha; 2)$ *for* $\mu = 2$ *and* $\alpha = 0.5; 1; 2; 3; 4$

3.2 *Density* $f(x:\mu;\alpha;3)$

From (8), (14) and the relation $B(\alpha;3)=\dfrac{2}{(\alpha+2)(\alpha+1)\alpha}$ it derives that

$$f(x:\mu;\alpha;3)=\begin{cases}\dfrac{(\alpha+1)(\alpha+2)}{4(\alpha+0.5)}\left\{\alpha\dfrac{x^{\alpha-1}}{\mu^{\alpha}}\right\}+\\ -\dfrac{\alpha(\alpha+2)}{4(\alpha+1.5)}\left\{(\alpha+1)\dfrac{x^{\alpha}}{\mu^{\alpha+1}}\right\}, & (0<x\le\mu)\\ \\ \dfrac{\alpha(\alpha+2)}{4(\alpha+0.5)}\left\{(\alpha+1)\mu^{\alpha+1}x^{-[(\alpha+1)+1]}\right\}+ & \\ & (\mu<x).\\ -\dfrac{\alpha(\alpha+1)}{4(\alpha+1.5)}\left\{(\alpha+2)\mu^{\alpha+2}x^{-[(\alpha+2)+1]}\right\}, & \end{cases} \tag{17}$$

In Figure 3 are given the graphs of the density $f(x:\mu;\alpha;\theta)$ for $\mu=2$ and $\alpha=1;2;3;4;5;6$.

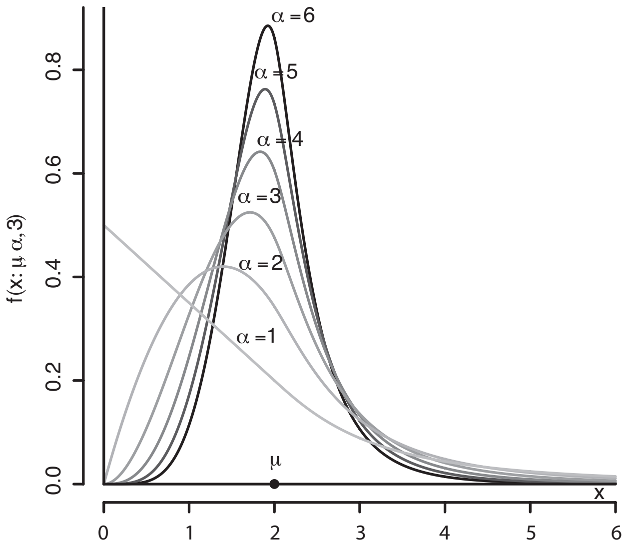

FIGURE 3. - *Graphs of* $f(x:\mu;\alpha;3)$ *for* $\mu=2$ *and* $\alpha=1;2;3;4;5;6$

3.3 *Density* $f(x:\mu;\alpha;4)$

From (8), (15) and from the relation $B(\alpha;4)=\dfrac{6}{(\alpha+3)(\alpha+2)(\alpha+1)\alpha}$, it follows that

$$f(x:\mu;\alpha;4)=\begin{cases} \dfrac{(\alpha+1)(\alpha+2)(\alpha+3)}{12(\alpha+0.5)}\left\{\alpha\dfrac{x^{\alpha-1}}{\mu^{\alpha}}\right\}+ & \\ -\dfrac{2}{12}\dfrac{\alpha(\alpha+2)(\alpha+3)}{12(\alpha+1.5)}\left\{(\alpha+1)\dfrac{x^{\alpha}}{\mu^{\alpha+1}}\right\}+ & (0<x\leq\mu) \\ +\dfrac{1}{12}\dfrac{\alpha(\alpha+1)(\alpha+3)}{(\alpha+2.5)}\left\{(\alpha+2)\dfrac{x^{\alpha+1}}{\mu^{\alpha+2}}\right\}, & \\ \dfrac{\alpha(\alpha+2)(\alpha+3)}{12(\alpha+0.5)}\left\{(\alpha+1)\mu^{\alpha+1}x^{-[(\alpha+1)+1]}\right\}+ & \\ -\dfrac{2\alpha(\alpha+1)(\alpha+3)}{12(\alpha+1.5)}\left\{(\alpha+2)\mu^{\alpha+2}x^{-[(\alpha+2)+1]}\right\}+ & (\mu<x). \\ +\dfrac{\alpha(\alpha+1)(\alpha+2)}{12(\alpha+2.5)}\left\{(\alpha+3)\mu^{\alpha+3}x^{-[(\alpha+3)+1]}\right\}, & \end{cases} \tag{18}$$

Figure 4 gives the graphs of the density $f(x:\mu;\alpha;4)$ for $\mu=2$ and $\alpha=1;2;3;4;5;6$.

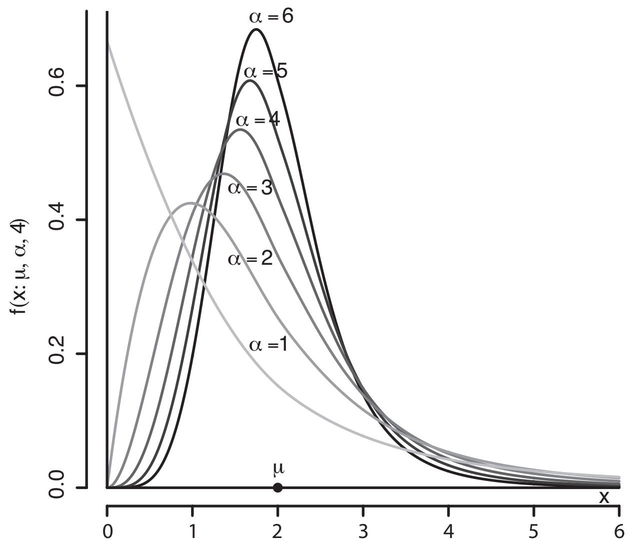

FIGURE 4. - *Graphs of* $f(x:\mu;\alpha;4)$ *for* $\mu=2$ *and* $\alpha=1;2;3;4;5;6$

4. EXAMPLES OF GRAPHS OF $f(x:\mu;\alpha;\theta)$ FOR θ NON INTEGER

It is obviously possible a ''PC'' evaluation of $f(x:\mu;\alpha;\theta)$ for non integer θ. In Figure 5 the graphs of $f(x:\mu;\alpha;4.66)$ for $\mu=2$ and $\alpha=1;2;2.2688;3;4;5$ are reported. The values $\theta=4.66$ and $\alpha=2.2688$ have been obtained[1] by using the mo-

[1] These results are due to Mariangela Zenga.

ments method on the individual income distribution derived from the 2006 Bank of Italy Survey on household income and wealth (Banca d'Italia, 2008).

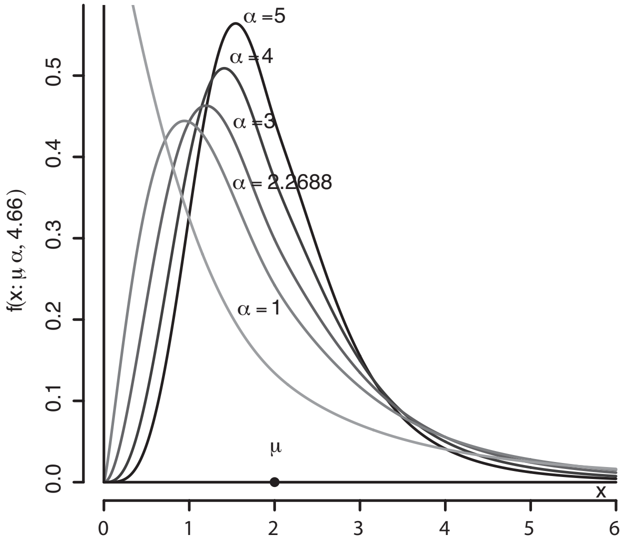

FIGURE 5. - *Graphs of* $f(x:\mu;\alpha;4.66)$ *for* $\mu = 2$ *and* $\alpha = 1; 2.2688; 3; 4; 5$

5. MOMENTS OF THE DENSITY $f(x:\mu;\alpha;\theta)$

Polisicchio (2008) has shown that the r-th moment around zero of the density $f(x:\mu;k)$ is given by

$$\mu_r' = \frac{\mu^r k^{1-r}}{(2r-1)} \frac{(1-k^{2r-1})}{(1-k)}, \quad (0 < k < 1; \mu > 0 \text{ and } r \in N) \tag{19}$$

where μ is the finite and positive expected value of the density $f(x:\mu;k)$. Consequently, the r-th moment $E(X^r)$ around zero of the density $f(x:\mu;\alpha;\theta)$ is obtained from

$$E(X^r) = \int_0^1 \frac{\mu^r k^{(1-r)}}{(2r-1)} \frac{(1-k^{2r-1})}{(1-k)} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)} dk\ . \tag{20}$$

In particular, for $r = 1$ we have

$$E(X) = \int_0^1 \mu \cdot \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)} = \mu\ . \tag{21}$$

Thus for the density $f(x:\mu;\alpha;\theta)$, the expectation is equal to the parameter μ.
From (20) it derives that

$$E(X^r) = \frac{\mu^r}{(2r-1)} \int_0^1 \frac{k^{\alpha-r}(1-k^{2r-1})(1-k)^{(\theta-1)-1}}{B(\alpha;\theta)} dk$$
$$= \frac{\mu^r}{(2r-1)} \frac{1}{B(\alpha;\theta)} \left\{ \int_0^1 k^{(\alpha-r+1)-1}(1-k)^{(\theta-1)-1} dk + \right.$$
$$\left. - \int_0^1 k^{(\alpha+r)-1}(1-k)^{(\theta-1)-1} dk \right\}. \tag{22}$$

The integral $\int_0^1 k^{(\alpha-r+1)-1}(1-k)^{(\theta-1)-1} dk$ exists for $r < (\alpha+1)$ and $\theta > 1$. This means that according to the (22) the r-th moment exists for $\theta > 1$ and $r < (\alpha+1)$. Consequently, from (22) it follows that

$$E(X^r) = \frac{\mu^r}{(2r-1)} \frac{1}{B(\alpha;\theta)} \{B(\alpha-r+1;\theta-1) - B(\alpha+r;\theta-1)\}, \tag{23}$$
$$(r \in N, r < \alpha+1, \theta > 1).$$

From (23) it derives that

$$E(X^2) = \frac{\mu^2}{3} \left\{ \frac{B(\alpha-1;\theta-1)}{B(\alpha;\theta)} - \frac{B(\alpha+2;\theta-1)}{B(\alpha;\theta)} \right\}.$$

Utilising the relation $B(a;b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, where $\Gamma(t)$ is the Gamma function, it derives that

$$E(X^2) = \frac{\mu^2}{3} \left\{ \frac{(\alpha+\theta-1)(\alpha+\theta-2)}{(\alpha-1)(\theta-1)} - \frac{(\alpha+1)\alpha}{(\alpha+\theta)(\theta-1)} \right\}$$
$$= \frac{\mu^2}{3(\theta-1)} \left\{ \frac{3\alpha^2(\theta-1) + 3\alpha(\theta-1)^2 + \theta(\theta^2-3\theta+2)}{(\alpha-1)(\alpha+\theta)} \right\}$$
$$= \frac{\mu^2}{3(\theta-1)} \left\{ \frac{3\alpha^2(\theta-1) + 3\alpha(\theta-1)^2 + \theta(\theta-1)(\theta-2)}{(\alpha-1)(\alpha+\theta)} \right\}$$
$$= \frac{\mu^2}{3} \left\{ \frac{3\alpha^2 + 3\alpha(\theta-1) + \theta(\theta-2)}{(\alpha-1)(\alpha+\theta)} \right\} \tag{24}$$

The variance of $f(x : \mu; \alpha; \theta)$ is given by

$$Var(X) = E(X^2) - \mu^2 = E(X^2) - \mu^2 \frac{3(\alpha-1)(\alpha+\theta)}{3(\alpha-1)(\alpha+\theta)}.$$

Now utilising the relation (24) and some calculations, it derives that

$$Var(X) = \frac{\mu^2\theta(\theta+1)}{3(\alpha-1)(\alpha+\theta)} \quad . \tag{25}$$

The third moment is given by

$$E(X^3) = \frac{\mu^3}{5}\left\{\frac{B(\alpha-2;\theta-1)}{B(\alpha;\theta)} - \frac{B(\alpha+3;\theta-1)}{B(\alpha;\theta)}\right\}$$
$$= \frac{\mu^3}{5(\theta-1)}\left\{\frac{(\alpha+\theta-1)(\alpha+\theta-2)(\alpha+\theta-3)}{(\alpha-1)(\alpha-2)} - \frac{(\alpha+2)(\alpha+1)\alpha}{(\alpha+\theta+1)(\alpha+\theta)}\right\}. \tag{26}$$

6. DISTRIBUTION FUNCTION F(X),$(0 < x \le \mu)$ AND MEDIAN OF THE DENSITY $f(x : \mu; \alpha; \theta)$

Polisicchio (2008) has shown that the distribution function of the density $f(x : \mu; k)$ is given by

$$P(X \le x) = \frac{1}{(1-k)}\left(1 - \left(\frac{\mu}{x}\right)^{0.5} k^{0.5}\right), \qquad \left(\mu k \le x \le \frac{\mu}{k}\right). \tag{27}$$

For $x = \mu$, the formula (27) gives

$$P(X \le \mu) = \frac{1}{(1-k)}(1 - k^{0.5}) = (1-k)^{-1} - k^{0.5}(1-k)^{-1}. \tag{28}$$

Consequently, for $x = \mu$ the cumulative probability $F(\mu)$ of the density $f(x : \mu; \alpha; \theta)$ is furnished by

$$F(\mu) = \int_0^1 \left\{(1-k)^{-1} - k^{0.5}(1-k)^{-1}\right\} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)} dk$$
$$= \frac{1}{B(\alpha;\theta)} \int_0^1 \left\{k^{\alpha-1}(1-k)^{\theta-2} - k^{\alpha+0.5-1}(1-k)^{\theta-2}\right\} dk$$
$$= \frac{1}{B(\alpha;\theta)}\{B(\alpha;\theta-1) - B(\alpha+0.5;\theta-1)\}$$
$$= \frac{1}{\theta-1}\left\{(\alpha+\theta-1) - \frac{\Gamma(\alpha+0.5)\Gamma(\alpha+\theta)}{\Gamma(\alpha)\Gamma(\alpha+\theta-0.5)}\right\}. \tag{29}$$

Lemma 2 $F(\mu) \ge \frac{1}{2}$, $(\alpha > 0; \theta > 0)$. *By (28), we note that, in the interval* $(0 < k < 1)$,

$$P[X \le \mu] = \frac{1}{(1-k)}(1 - k^{0.5}) = \frac{1}{1+\sqrt{k}}$$

is a decreasing function equal to 1 *for* $k = 0$ *and equal to* 0.5 *for* $k \to 1$. *Consequently, its weighted arithmetic mean with weights* $\frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)}$ *must assume a value in the interval* $(0.5; 1)$. *In other words,* $\frac{1}{2} \le F(\mu) \le 1$. *Table 1 gives* $F(\mu)$ *for some values of* θ *and* α.

TABLE 1. - *Cumulative distribution function $F(\mu)$ for some values of the parameters α and θ*

θ/α	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**
1.5	0.7267607	0.6438055	0.6046945	0.5821353	0.5675113	0.5572807	0.5497297	0.5439310
2	0.7500000	0.6666667	0.6250000	0.6000000	0.5833333	0.5714286	0.5626000	0.5555556
2.5	0.7674491	0.6849190	0.6418778	0.6152749	0.5971472	0.5839803	0.5739740	0.5661080
3	0.7812500	0.7000000	0.6562500	0.6285714	0.6093750	0.5952381	0.5843750	0.5757580
3.5	0.7925633	0.7127766	0.6687162	0.6403105	0.6203203	0.6054270	0.5938739	0.5846370
4	0.8102083	0.7238095	0.6796875	0.6507937	0.6302083	0.6147186	0.6026042	0.5928520
4.5	0.8102558	0.7334812	0.6894573	0.6602440	0.6392106	0.6232471	0.6106722	0.6004870
5	0.8173828	0.7420635	0.6982422	0.6688312	0.6474609	0.6311189	0.6181641	0.6076150
5.5	0.8236778	0.7497561	0.7062062	0.6766871	0.6550653	0.6384202	0.6251503	0.6142920

From Lemma 2 it derives that $x_{(0.5)} \leq \mu$, where $x_{(0.5)}$ denotes the median of $f(x : \mu; \alpha; \theta)$. Consequently, for the calculation of $x_{(0.5)}$ it is necessary to obtain the mathematical expression of $F(x)$ for $0 < x \leq \mu$. From (27) and from Figure 1, it follows that for $0 < x \leq \mu$:

$$
\begin{aligned}
F(x) &= \int_0^{\frac{x}{\mu}} \left\{(1-k)^{-1} - \left(\frac{\mu}{x}\right)^{0.5} (1-k)^{-1} k^{0.5}\right\} \frac{k^{\alpha-1}(1-k)^{\theta-1}}{B(\alpha;\theta)} dk \\
&= \frac{1}{B(\alpha;\theta)} \left\{ \int_0^{\frac{x}{\mu}} k^{\alpha-1}(1-k)^{\theta-2} dk - \int_0^{\frac{x}{\mu}} \left(\frac{\mu}{x}\right)^{0.5} k^{\alpha+0.5-1}(1-k)^{\theta-2} dk \right\} \\
&= \frac{B(\alpha;\theta-1)}{B(\alpha;\theta)} B\left(\frac{x}{\mu};\alpha;\theta-1\right) + \\
&\quad - \left(\frac{\mu}{x}\right)^{0.5} \frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} B\left(\frac{x}{\mu};\alpha+0.5;\theta-1\right).
\end{aligned} \tag{30}
$$

From (30) it is possible to get the median $x_{(0.5)}$ by solving the equation:

$$
\begin{aligned}
&\frac{B(\alpha;\theta-1)}{B(\alpha;\theta)} B\left(\frac{x}{\mu};\alpha;\theta-1\right) + \\
&- \frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} \left(\frac{\mu}{x}\right)^{0.5} B\left(\frac{x}{\mu};\alpha+0.5;\theta-1\right) = 0.5.
\end{aligned} \tag{31}
$$

Note that if $0 < \theta \leq 1$ the median can be obtained working on the density (8).

In Table 2 the tabulation of the function $F(X)$ for $(\mu = 1; \alpha = 2; \theta = 2)$ and for $(\mu = 1; \alpha = 2.5; \theta = 4.5)$ is reported $(0 < x \leq \mu)$.

From Table 2 it derives that for $f(x : 1; 2; 2)$ the median is equal to 0.913 and for $f(x; 1; 2.5; 4.5)$ the median is 0.782.

TABLE 2. - *Distribution function* $F(x)$ *for* $0 < x \leq 1$ *and for the parameters* $(\mu = 1; \alpha = 2; \theta = 2)$ *and* $(\mu = 1; \alpha = 4.5; \theta = 4.5)$

x	$F(x; \mu = 1; \alpha = 2; \theta = 2)$	$F(x; \mu = 1; \alpha = 2,5; \theta = 4,5)$
0.010	0.0001	0.0000
0.050	0.0015	0.0016
0.100	0.0060	0.0086
0.150	0.0135	0.0220
0.200	0.0240	0.0420
0.250	0.0375	0.0682
0.300	0.0540	0.0999
0.350	0.0735	0.1360
0.400	0.0960	0.1758
0.450	0.1215	0.2181
0.500	0.1500	0.2619
0.550	0.1815	0.3064
0.600	0.2160	0.3507
0.650	0.2535	0.3941
0.700	0.2940	0.4359
0.750	0.3375	0.4758
0.782	0.3669	0.5001
0.800	0.3840	0.5134
0.850	0.4335	0.5485
0.900	0.4860	0.5811
0.913	0.5001	0.5892
0.950	0.5415	0.6113
0.990	0.5880	0.6338
1.000	0.6000	0.6392

7. MOMENTS OBTAINED (DIRECTLY) FROM THE DENSITY $f(x : \mu; \alpha; \theta)$

In some statistical analyses, it is necessary to evaluate the expectation of X^r for $X \leq \mu$ or for $X > \mu$:

$$E(X^r | X \leq \mu) = \frac{\int_0^\mu t^r f(t) dt}{F(\mu)};$$

$$E(X^r | X > \mu) = \frac{\int_\mu^\infty t^r f(t) dt}{1 - F(\mu)}.$$

For simplicity's sake it is possible to put $\mu = 1$. In this case, the density

$f(x:\mu;\alpha;\theta)$ is given by

$$f(x:1;\alpha;\theta)=\begin{cases}\dfrac{x^{-1.5}}{2B(\alpha;\theta)}\displaystyle\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & (0<x<1)\\ \dfrac{x^{-1.5}}{2B(\alpha;\theta)}\displaystyle\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk, & (1<x).\end{cases} \tag{32}$$

Consequently, $E(X^r)$ is given by the sum:

$$\int_0^1\left\{\frac{x^r x^{-1.5}}{2B(\alpha;\theta)}\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right\}dx+$$

$$+\int_1^\infty\left\{\frac{x^r x^{-1.5}}{2B(\alpha;\theta)}\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right\}dx$$

$$\Downarrow$$

$$E(X^r)=\frac{1}{2B(\alpha;\theta)}\int_0^1\left(\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right)d\left(\frac{x^{r-0.5}}{r-0.5}\right)+ \tag{33}$$

$$+\frac{1}{2B(\alpha;\theta)}\int_1^\infty\left(\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right)d\left(\frac{x^{r-0.5}}{r-0.5}\right). \tag{34}$$

Using integration by parts, the outer integral in (33) is given by

$$\left(\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right)\frac{x^{r-0.5}}{r-0.5}\Bigg|_0^1-\int_0^1\left(\frac{x^{r-0.5}}{r-0.5}\right)\left[x^{\alpha+0.5-1}(1-x)^{\theta-2}\right]dx$$

$$=$$

$$\left\{\int_0^1 k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\frac{1}{r-0.5}-\lim_{x\to 0}\left(\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\frac{x^{r-0.5}}{r-0.5}\right)\right\}+$$

$$-\frac{1}{(r-0.5)}\int_0^1 x^{\alpha+r-1}(1-x)^{\theta-2}dx. \tag{35}$$

For $r>0.5$, $\lim_{x\to 0}\left(\int_0^x k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\,x^{r-0.5}\right)=0$. Consequently, the value of (35) is given by

$$\frac{1}{(r-0.5)}\{B(\alpha+0.5;\theta-1)-B(\alpha+r;\theta-1)\}. \tag{36}$$

In conclusion, for $r>0.5$

$$\int_0^1 x^r f(x:1;\alpha;\theta)dx=\frac{1}{2(r-0.5)}\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+r;\theta-1)}{B(\alpha;\theta)}\right\}. \tag{37}$$

Using integration by parts, the integral in (34) is given by:

$$\left(\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right)\frac{x^{r-0.5}}{r-0.5}\Bigg|_1^{\infty}+$$

$$-\int_1^{\infty}\left(\frac{x^{r-0.5}}{r-0.5}\right)\left(\left(\frac{1}{x}\right)^{\alpha+0.5-1}\left(1-\frac{1}{x}\right)^{\theta-2}(-1)x^{-2}\right)dx=$$

$$=\frac{1}{(r-0,5)}\left\{\lim_{x\to\infty}\left(\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\frac{1}{x^{-r+0.5}}\right)-\int_0^1 k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right\}+$$

$$+\frac{1}{(r-0.5)}\int_1^{\infty}\left(\frac{1}{x}\right)^{\alpha-r+2}\left(1-\frac{1}{x}\right)^{\theta-2}dx. \tag{38}$$

It is evident that for $r > 0.5$ the

$$\lim_{x\to\infty}\left(\int_0^{\frac{1}{x}} k^{\alpha+0.5-1}(1-k)^{\theta-2}dk\right)\cdot\frac{1}{x^{-r+0.5}}$$

is an indeterminate form of type $\frac{0}{0}$. Then, applying the De l'Hopital rule, it derives that

$$\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)^{\alpha+0.5-1}\left(1-\frac{1}{x}\right)^{\theta-2}(-1)\left(\frac{1}{x}\right)^2}{(-r+0.5)x^{-r-0.5}}=$$

$$=-\frac{1}{(-r+0.5)}\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{\theta-2}\frac{1}{x^{\alpha+0.5-1+2-r-0.5}}$$

$$=-\frac{1}{(-r+0.5)}\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{\theta-2}\frac{1}{x^{\alpha+1-r}}=0\ ,\qquad \text{for } (\alpha+1)>r.$$

Moreover, putting $y=\frac{1}{x}$ the integral $\int_1^{\infty}\left(\frac{1}{x}\right)^{\alpha-r+2}\left(1-\frac{1}{x}\right)^{\theta-2}dx$ is equal to

$$\int_1^0 y^{\alpha-r+2}(1-y)^{\theta-2}(-1)y^{-2}dy=\int_0^1 y^{\alpha-r}(1-y)^{\theta-2}dy$$
$$=B(\alpha-r+1;\theta-1).$$

In conclusion, for $(\alpha+1)>r$

$$\int_1^{\infty}x^rf(x:1;\alpha;\theta)dx=\frac{1}{B(\alpha;\theta)}\frac{1}{2(r-0.5)}\{-B(\alpha+0.5;\theta-1)+$$
$$+B(\alpha-r+1;\theta-1)\}.$$

Then, the moment $E(X^r)$ of the density $f(x:1;\alpha;\theta)$ is given by

$$E(X^r) = \frac{1}{2(r-0.5)}\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+r;\theta-1)}{B(\alpha;\theta)}\right\}+ \\ +\frac{1}{2(r-0.5)}\left\{\frac{B(\alpha-r+1;\theta-1)-B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\}. \quad (39)$$

Now, it is well known that if $Y=\mu X$, $\mu>0$, then

$$E(Y^r)=\mu^r E(X^r) \quad (40)$$

Then from Lemma 1, (39) and (40), it derives that the r-th moment of the density $f(y:\mu;\alpha;\theta)$ is given by

$$E(Y^r) = \frac{\mu^r}{2(r-0.5)}\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+r;\theta-1)}{B(\alpha;\theta)}\right\}+ \\ +\frac{\mu^r}{2(r-0.5)}\left\{\frac{B(\alpha-r+1;\theta-1)-B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\}. \quad (41)$$

Obviously, formula (41) coincides with formula (23), but formula (41) is more informative than formula (23) because the value of the r-th moment can be split as follows

$$\begin{aligned} E(Y^r) &= \int_0^{\mu} y^r f(y:\mu;\alpha;\theta)dy + \int_{\mu}^{\infty} y^r f(y:\mu;\alpha;\theta)dy \\ &= E(Y^r/Y\le\mu)F(\mu)+E(Y^r/Y>\mu)(1-F(\mu)) \\ &= \frac{\mu^r}{2(r-0.5)}\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+r;\theta-1)}{B(\alpha;\theta)}\right\}+ \\ &\quad +\frac{\mu^r}{2(r-0.5)}\left\{\frac{B(\alpha-r+1;\theta-1)-B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\}. \end{aligned} \quad (42)$$

8. MEAN DEVIATION

The mean deviation of a continuous random variable X with expectation μ and density $f(x)$ is given by

$$\begin{aligned} E(|X-\mu|) &= \int_0^{\mu}(\mu-x)f(x)dx + \int_{\mu}^{\infty}(x-\mu)f(x)dx \\ &= \mu F(\mu) - \int_0^{\mu} xf(x)dx + \int_{\mu}^{\infty} xf(x)dx - \mu[1-F(\mu)]. \end{aligned} \quad (43)$$

From (42) it derives that in the case of the density $f(x:\mu;\alpha;\theta)$:

$$\int_0^{\mu} xf(x:\mu;\alpha;\theta)dx = \mu\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+1;\theta-1)}{B(\alpha;\theta)}\right\}; \quad (44)$$

$$\int_{\mu}^{\infty} xf(x:\mu;\alpha;\theta)dx = \mu\left\{\frac{B(\alpha;\theta-1)-B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\}. \tag{45}$$

The term in the curly brackets of (45) is the expression of $F(\mu)$, see formula (29).

$$\begin{aligned} F(\mu) &= \left\{\frac{B(\alpha;\theta-1)}{B(\alpha;\theta)} - \frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\} \\ &= \left\{\frac{\alpha}{\theta-1} + 1 - \frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\}. \end{aligned} \tag{46}$$

From (46) it follows that

$$[1-F(\mu)] = \left\{\frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} - \frac{\alpha}{\theta-1}\right\}. \tag{47}$$

The term in curly brackets of (44) is equal to $[1-F(\mu)]$. In effect

$$\begin{aligned} \left\{\frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} - \frac{B(\alpha+1;\theta-1)}{B(\alpha;\theta)}\right\} &= \left\{\frac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} - \frac{\alpha}{(\theta-1)}\right\} \\ &= [1-F(\mu)]. \end{aligned} \tag{48}$$

From the results (46) and (48), the expression (43) becomes:

$$\begin{aligned} E(|X-\mu|) &= \mu F(\mu) - \mu[1-F(\mu)] + \mu F(\mu) - \mu[1-F(\mu)] \\ &= 2\mu F(\mu) - 2\mu[1-F(\mu)] \\ &= 2\mu[F(\mu) - 1 + F(\mu)] \\ &= 2\mu[2F(\mu) - 1]. \end{aligned} \tag{49}$$

The relative mean deviation P of the density $f(x:\mu;\alpha;\theta)$ is given by

$$P = \frac{E(|X-\mu|)}{2\mu} = \frac{2\mu[2F(\mu)-1]}{2\mu} = [2F(\mu)-1]. \tag{50}$$

9. INEQUALITY MEASURE $A(\mu)$

In Section 8 it has been shown that for the density $f(x:\mu;\alpha;\theta)$

$$\begin{aligned} \int_{0}^{\mu} xf(x:\mu;\alpha;\theta)dx &= \mu\left\{\frac{B(\alpha+0.5;\theta-1)-B(\alpha+1;\theta-1)}{B(\alpha;\theta)}\right\} \\ &= \mu(1-F(\mu)); \end{aligned} \tag{51}$$

$$\begin{aligned} \int_{\mu}^{\infty} xf(x:\mu;\alpha;\theta)dx &= \mu\left\{\frac{B(\alpha;\theta-1)-B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}\right\} \\ &= \mu F(\mu). \end{aligned} \tag{52}$$

From (51) and (52) it derives that for the density $f(x:\mu;\alpha;\theta)$:

$$E(X|X \leq \mu) = \frac{\mu[1-F(\mu)]}{F(\mu)}; \tag{53}$$

$$E(X|X > \mu) = \frac{\mu F(\mu)}{[1-F(\mu)]}. \tag{54}$$

Consequently, for the density $f(x:\mu;\alpha;\theta)$, the Zenga's (2007) point inequality measure $A(x)$, for $x=\mu$, is equal to:

$$A(\mu) = 1 - \frac{E(X|X \leq \mu)}{E(X|X > \mu)} = 1 - \frac{\mu\dfrac{[1-F(\mu)]}{F(\mu)}}{\mu\dfrac{F(\mu)}{[1-F(\mu)]}} = 1 - \left(\frac{[1-F(\mu)]}{F(\mu)}\right)^2. \tag{55}$$

10. UNEXPECTED EQUALITIES

From (51) and (52) it derives that:

$$\begin{cases} \displaystyle\int_0^\mu \frac{x}{\mu} f(x:\mu;\alpha;\theta)dx = 1-F(\mu), \\ \displaystyle\int_\mu^\infty \frac{x}{\mu} f(x:\mu;\alpha;\theta)dx = F(\mu). \end{cases}$$

Now, from the mathematical expression of $F(\mu)$ and $(1-F(\mu))$ it is possible to get, in a different way the following known equality

$$B(\alpha;\theta-1) = B(\alpha+1;\theta-1) + B(\alpha;\theta).$$

PROOF:
From (48) and (46) it derives that

$$\begin{cases} [1-F(\mu)] & = \dfrac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)} - \dfrac{B(\alpha+1;\theta-1)}{B(\alpha;\theta)}, \\ F(\mu) & = \dfrac{B(\alpha;\theta-1)}{B(\alpha;\theta)} - \dfrac{B(\alpha+0.5;\theta-1)}{B(\alpha;\theta)}, \end{cases}$$

$$\Downarrow$$

$$1 = \frac{B(\alpha;\theta-1)}{B(\alpha;\theta)} - \frac{B(\alpha+1;\theta-1)}{B(\alpha;\theta)} \rightarrow$$

$$B(\alpha;\theta-1) = B(\alpha+1;\theta-1) + B(\alpha;\theta).$$

11. CONCLUSIONS

The new three parameter density function $f(x:\mu;\alpha;\theta)$ proposed in this paper, has some interesting characteristics that can be useful for income and wealth, as well as for actuarial and financial distributions.

First of all according to the graphs of the density $f(x:\mu;\alpha;\theta)$ it seems that the shapes of the new density function are broader than those of the more traditional models for income distributions like Dagum's distribution. Then for all the admissible values of the three parameters $(\mu > 0; \alpha > 0; \theta > 0)$ the expectation of the new r.v. is finite and is equal to μ. Moreover, the new density has a positive asymmetry because $F(\mu) \geq \frac{1}{2}$, where $F(x)$ is the distribution function. This paper shows that the r-th moment exists if $r < (\alpha + 1)$. The upper and lower moments (calculated for $x = \mu$), as well as the ordinary ones are simple expressions of the beta functions whose parameters are α, θ and the degree r of the moments. Moreover, the variance is equal to $\frac{\mu^2}{3} \frac{\theta(\theta+1)}{[\alpha^2 + \alpha(\theta-1) - \theta]}$, the mean deviation is equal to $2\mu[2F(\mu) - 1]$.

''Simple'' expressions for moments and distribution function, related to the new three parameters random variable with density $f(x:\mu;\alpha;\theta)$, have been derived in the case $\theta > 1$. It seems that a different approach is necessary to handle the case $0 < \theta \leq 1$.

Some estimations of the parameters $(\mu;\alpha;\theta)$ have been obtained by utilizing the moment method, the likelihood method and some other non-traditional methods; the distributions we have employed have been derived from the 2006 Bank of Italy sample survey on Household income and wealth (Banca d'Italia, 2008).

RIASSUNTO

In questo articolo si ricava, per variabili casuali non negative X, una nuova funzione di densità di probabilità $f(x:\mu;\alpha;\theta)$, $(\mu > 0; \alpha > 0; \theta > 0)$. *La nuova densità è ottenuta come miscuglio di particolari variabili casuali Pareto troncate recentemente analizzate da M. Polisicchio. Il parametro* $\mu = E(X)$. *Inoltre la nuova densità presenta asimmetria positiva e coda-destra di tipo Paretiano. La varianza è uguale a* $\frac{\mu^2}{3} \cdot \frac{\theta(\theta+1)}{[\alpha^2 + \alpha(\theta-1) - \theta]}$. *I momenti ed i momenti troncati in* $x = \mu$ *sono espressioni compatte di funzioni Beta.*

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