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THE DISTRIBUTION MODEL WITH LINEAR INEQUALITY CURVE $I(p)$

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SUMMARY

The aim of an important branch of inequality analysis is the investigation into the features of the distribution models with specific kinds of inequality curves. This paper is an investigation into the distribution model with linear inequality $I(p)$ curve. The definition of the corresponding distribution function, and a procedure to obtain the probability density function are described. An analysis of the constraints that the parameters of the line must satisfy is also provided. The methodological results are supported by two applications with real air traffic control data.

Keywords: *Inequality, Lorenz $L(p)$ Curve, Zenga $I(p)$ Curve, Income Distributions.*

1. INTRODUCTION

A significant number of papers have been dedicated to the analysis of the features of some inequality-evaluating tools. Not surprisingly, the most studied are Lorenz curve $L(p)$ and Gini concentration ratio. Nevertheless, there are other tools that need further investigation to reveal their individual properties. The present work falls into this category, where it chiefly analyses the inequality $I(p)$ curve and the inequality index I proposed by Zenga (2007). The $I(p)$ curve is defined as one minus the ratio between lower and upper mean:

$$I(p) = 1 - \frac{\overset{-}{M}_{(p)}}{\overset{+}{M}_{(p)}} \qquad p \in (0,1),$$

where the lower and the upper means of the non-negative continuous random variable X (with finite and positive expectation μ) are

$$\overset{-}{M}_{(p)} = \frac{1}{p}\int_0^p F_X^{-1}(y)dy, \qquad p \in (0,1)$$

and

$$\overset{+}{M}_{(p)} = \frac{1}{1-p}\int_p^1 F_X^{-1}(y)dy \qquad p \in (0,1)$$

respectively, F_X being the distribution function of X. Starting from $I(p)$ curve, the

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inequality index I, defined by

$$I = \int_0^1 I(p)dp$$

can be obtained (see Zenga, 2007).

The aim of this paper is to investigate the features of the distribution model that generates a specific $I(p)$ curve. The first person to explore the subject was Polisicchio (2008). Here, the author provides the distribution model that generates a uniform inequality $I(p)$ curve. In particular it is proved that, for a fixed $k \in]0,1[$, the continuous random variable X has $I(p)$ curve given by

$$I_X(p) = k, \qquad \forall p \in (0,1)$$

if and only if the distribution function of X is:

$$F_X(x) = \begin{cases} 0 & x < \mu k \\ \dfrac{1}{1-k}[1 - \sqrt{\dfrac{\mu k}{x}}] & \mu k \leq x \leq \mu/k \\ 1 & x > \mu/k \end{cases} \tag{1}$$

and therefore the probability density function is:

$$f_X(x) = \begin{cases} \dfrac{\sqrt{\mu k}}{2(1-k)} x^{-1,5} & \mu k \leq x \leq \mu/k \\ 0 & otherwise \end{cases}.$$

In this case, μ is the expectation, and k is the uniformity level (or conversely $1-k$, the inequality level) of the random variable X.

This article generalizes the previous result, and the distribution model generating a linear $I(p)$ curve is studied. The paper is organized as follows: Section 2 examines the distribution function with $I(p)$ curve satisfying $I(p) = ap + b,\ (a,b) \in \mathbb{R};$ in Section 3 the features of the two parameters of the line (a and b) are investigated; Section 4 is dedicated to the probability density function, and section 5 consists of an assessment of the linear case, while in Section 6 two applications with real data are described. The final section contains some brief remarks.

2. THE CASE OF THE STRAIGHT LINE

Taking the previous result for the uniform $I(p)$ curve as a point of departure, a generalization is provided in this section. The first natural extension is to investigate the distribution model with linear $I(p)$ curve.

Firstly, for simplification (avoiding loss of generality), the $I(p)$ curve will be considered defined on the interval $[0,1]$ (0 and 1 included) in the following. This extension will not cause any difficulty, since the $I(p)$ curve of the non-negative continuous random variable X can be continuously extended to the extreme values of the interval $[0, 1]$, setting:

$$I(0) := \lim_{p \to 0^+} I(p) = 1 - \frac{x^-}{\mu}$$

and

$$I(1) := \lim_{p \to 1^-} I(p) = 1 - \frac{\mu}{x^+},$$

where x^- is the lower and x^+ the upper value of the support of X, and μ the expectation (with the agreement that if the upper bound of the support is not finite, then $I(1) = 1$). For further details about the extension of $I(p)$, see Polisicchio (2008). Thus, in the following, the inequality $I(p)$ curve is a continuous function, defined on the closed interval $[0, 1]$.

The next theorem states the analytical form of the distribution function generating a linear $I(p)$ curve.

Theorem 1

Let X be a non-negative continuous random variable, with finite and positive expectation μ, and with $I(p)$ curve given by

$$I(p) = ap + b \qquad \forall p \in (0, 1) \quad \text{with } a \in \mathbb{R} \text{ and } b \in \mathbb{R} \tag{2}$$

then the distribution function F of X satisfies:

$$x = \mu \cdot \frac{aF^2(x) - 2aF(x) - b + 1}{\left[-aF^2(x) - bF(x) + 1\right]^2}.$$

Proof:

By the definition of $I(p)$ curve, if (2) holds, then:

$$\frac{\overset{-}{M}(p)}{\overset{+}{M}(p)} = 1 - ap - b, \qquad \forall p \in (0, 1). \tag{3}$$

By the identity

$$p\overset{-}{M}(p) + (1-p)\overset{+}{M}(p) = \mu, \qquad \forall p \in (0, 1) \tag{4}$$

it follows that

$$\overset{+}{M}(p) = \frac{\mu - p\overset{-}{M}(p)}{1-p},$$

and therefore equation (3) becomes

$$\overset{-}{M}(p) = \frac{[\mu - p\overset{-}{M}(p)](1 - ap - b)}{1-p},$$

that is:

$$\bar{M}_{(p)} = \mu \cdot \left[\frac{-ap - b + 1}{-ap^2 - bp + 1}\right].$$

By the definition of the lower mean, the last equation becomes

$$\int_0^p F^{-1}(t)dt = \mu p \cdot \left[\frac{-ap - b + 1}{-ap^2 - bp + 1}\right]$$

and, deriving with respect to p:

$$F^{-1}(p) = \mu \cdot \frac{ap^2 - 2ap - b + 1}{[-ap^2 - bp + 1]^2}.$$

Finally, recalling the transformation $F(x) = p$ and its inverse $F^{-1}(p) = x$, it holds that:

$$x = \mu \cdot \frac{aF^2(x) - 2aF(x) - b + 1}{[-aF^2(x) - bF(x) + 1]^2} \tag{5}$$

or equivalently

$$x(p) = \mu \cdot \frac{ap^2 - 2ap - b + 1}{[-ap^2 - bp + 1]^2}. \tag{6}$$

For some values of a and b, equation (5) can implicitly define the distribution function F of a random variable with linear inequality $I(p)$ curve satisfying (2).

It is easy to see that it is not possible, for all values of a and b, rearranging (5) to have an explicit definition of F. However, for particular values of the parameters, this can be done. For example, setting $a = 0$ and $b = 1 - k$ and rearranging, equation (5) becomes

$$1 - (1 - k)F(x) = \sqrt{\frac{\mu k}{x}}$$

and therefore it follows that:

$$F(x) = \frac{1}{1 - k}\left[1 - \sqrt{\frac{\mu k}{x}}\right]. \tag{7}$$

The function F defined in (7) takes on values between 0 and 1 if and only if $x \in [\mu k, \mu/k]$: since it has to be a distribution function it must be equal to 0 if $x < \mu k$ and to 1 if $x > \mu/k$. It is not surprising that F is the distribution function with the uniform $I(p)$ curve, already defined in (1).

In order to have an idea of the behaviour of F in (5), in Figure 1 and in Figure 2, some graphs are shown. Each of these corresponds to a fixed couple of values for the parameters a and b.

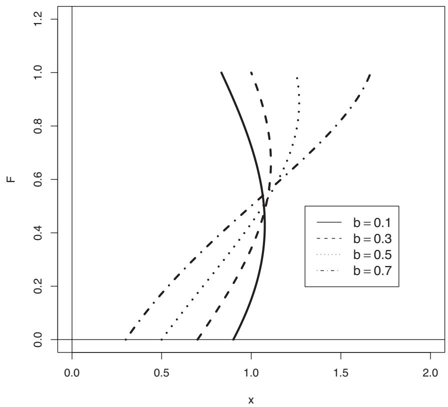

FIGURE 1. - *Graphs of F for some values of the parameter b, with* $a < 0$

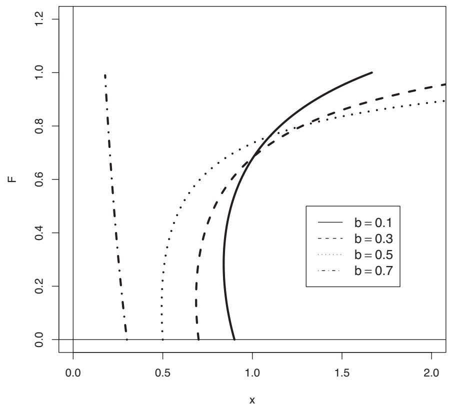

FIGURE 2. - *Graphs of F for some values of the parameter b, with* $a > 0$

Two important points arise from the figures: the first one is that for some values of the parameters a and b, F is not a well-defined function; the second one is that for some values of the parameters, F is not a monotonically increasing function. Obviously, all the couples (a, b) which do not define by (5) a well-defined increasing function F must be discarded. In the following section, an analysis of the conditions that a and b must satisfy, is provided.

3. THE CONSTRAINTS OF THE PARAMETERS

As shown before, equation (5) can implicitly define a distribution function F. Obviously, it depends on the values of the two parameters a and b. Since the inequality curve $I(p)$ can take on values between 0 and 1, if (2) holds, it is easy to verify that only the couples (a, b) lying in the region Ω of the plan (A, B) defined by:

$$\begin{cases} 0 \leq b \leq 1 \\ b < 1 - a \end{cases}$$

can be considered. The couples $(a, 1 - a)$ are discarded, since by choosing $F(x) = p = 1$ the denominator of (5) is zero. To simplify the following, Ω can be partitioned into two regions:

$$\Omega_1 = \{(a, b) \in \Omega \ : \ a \geq 0\}$$

$$\Omega_2 = \{(a, b) \in \Omega \ : \ a < 0\}.$$

Obviously, by definition, it holds that:

$$\Omega_1 \cap \Omega_2 = \emptyset;$$

$$\Omega_1 \cup \Omega_2 = \Omega.$$

With the aim of finding the restrictions that the parameters a and b must satisfy in order to define a distribution function F in (5), the derivative with respect to p of equation (6) is obtained. In this way, after a rearrangement, the following ratio is achieved

$$\frac{dx(p)}{dp} = \frac{N(p)}{D(p)} \tag{8}$$

where the numerator is

$$N(p) = 2\mu[a^2p^3 - 3a^2p^2 + 3ap(1-b) - a + b - b^2] \tag{9}$$

and the denominator is

$$D(p) = [-ap^2 - bp + 1]^3. \tag{10}$$

Now, the behavior of the two functions $N = N(p)$ and $D = D(p)$ is examined separately, in order to study the positivity of function $dx(p)/dp$.

Firstly, consider numerator function N. If $(a, b) \in \Omega_1$ then N is not decreasing for $p \in [0, 1]$. Therefore it follows that $N(p) > 0$, $\forall p \in [0, 1]$ if, and only if, $N(0) > 0$, and this is equivalent to $-a + b - b^2 > 0$, since the trivial case $\mu = 0$ can be discarded.

On the other hand, if $(a, b) \in \Omega_2$, the function N does not increase for $p \in [0, 1]$. Then, similarly as before, $N(p) > 0$, $\forall p \in [0, 1]$ if, and only if, $N(1) > 0$ and that happens (with μ positive) if and only if $2a + b > 0$.

In conclusion, the results of the investigation regarding the function N, numerator of the derivative of $x(p)$, can be summarized as follows:

- if $(a, b) \in \Omega_1$: $\quad N(p) > 0,\ \forall p \in [0, 1] \Leftrightarrow -a + b - b^2 > 0$
- if $(a, b) \in \Omega_2$: $\quad N(p) > 0,\ \forall p \in [0, 1] \Leftrightarrow 2a + b > 0.$

Through an analysis of the behavior of denominator function D, it can be proved that

$$\forall (a, b) \in \Omega: \qquad D(p) > 0,\ \forall p \in [0, 1].$$

Matching the restrictions for both N and D, the function

$$x(p) = \mu \cdot \frac{ap^2 - 2ap - b + 1}{[-ap^2 - bp + 1]^2}$$

is monotonically increasing if, and only if, the following conditions hold:

$$\text{for } (a, b) \in \Omega_1 \qquad -a + b - b^2 > 0 \tag{11}$$

and

$$\text{for } (a, b) \in \Omega_2 \qquad 2a + b > 0. \tag{12}$$

Since a function is monotonically increasing if, and only if, its inverse is monotonically increasing, and since $x = x(p)$ is the inverse of $F = F(x)$, the constraints (11) and (12) lead to the identification of the area consisting of only the couples (a, b) in the plan (A, B), for which the equation

$$I(p) = ap + b \qquad \forall p \in [0, 1]$$

can be verified, because the generating function F is the distribution function of a continuous random variable. Figure 3 shows the area considered: an analytical parametrization is given by:

$$\begin{cases} -\frac{b}{2} < a < b - b^2 \\ 0 < b < 1 \end{cases} . \qquad (13)$$

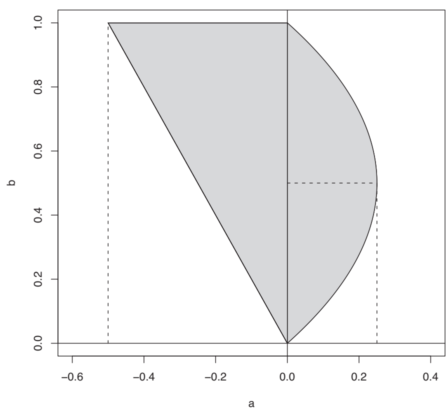

FIGURE 3. - *The region of the acceptable couples of the parameters* (a, b)

4. THE PROBABILITY DENSITY FUNCTION

At this point, the implicit expression of the distribution function with linear $I(p)$ curve and the constraints of the line parameters a and b have been investigated. The aim of this section is to give an idea about the probability density function that generates a linear $I(p)$ curve. Indeed it is not possible to have the explicit expression of the probability density function, but it can be ''indirectly'' obtained by the following procedure.

Let a^* and b^* be two values satisfying the conditions (13), and let X be a continuous non-negative random variable with linear $I(p)$ curve:

$$I_X(p) = a^* p + b^* \qquad \forall p \in (0, 1).$$

Consider a fixed $p \in (0, 1)$. The corrisponding quantile x of X can be calculated by (6). Since the quantile function (6) is the inverse of the distribution function, it holds that:

$$\frac{d}{dp}F[x(p)] = \frac{1}{\frac{d}{dp}[F^{-1}(p)]}.$$

Now, $F^{-1}(p)$ can be replaced by $x(p)$, and $\frac{d}{dp}F[x(p)]$ can be viewed as the probability density function of X depending on p.

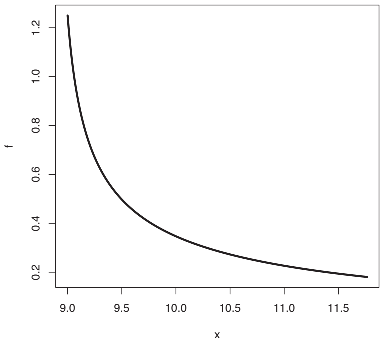

FIGURE 4. - *The probability density function with* $I(p) = 0,05p + 0,1$ *and* $\mu = 10$

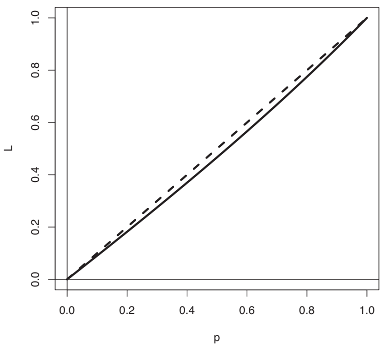

FIGURE 5. - $L(p)$ *curve of the probability density function of Figure 4*

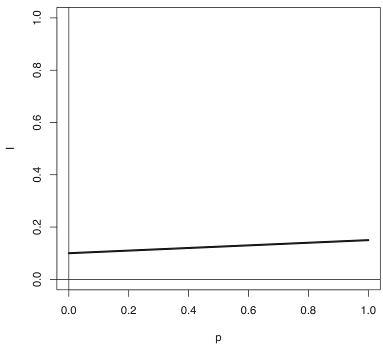

FIGURE 6. - *Linear* $I(p)$ *curve of the probability density function of Figure 4*

Therefore, for any $p \in (0, 1)$, it is possible to compute the value of the probability density function of the random variable X, calculated in the corresponding quantile $x = x(p)$. Using the notation of (8), (9) and (10), it holds that:

$$\frac{d}{dp}F[x(p)] = \frac{1}{\frac{d}{dp}[x(p)]} = \frac{D(p)}{N(p)} = \frac{[-a^*p^2 - b^*p + 1]^3}{2\mu[(a^*)^2p^3 - 3(a^*)^2p^2 + 3a^*p(1-b^*) - a^* + b^* - (b^*)^2]}.$$

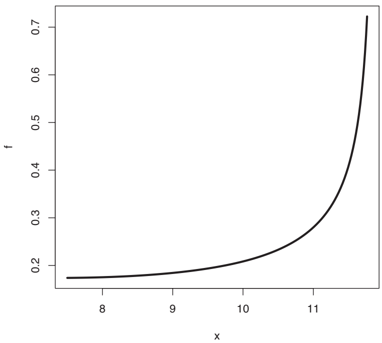

FIGURE 7. - *The probability density function with* $I(p) = -0,1p + 0,25$ *and* $\mu = 10$

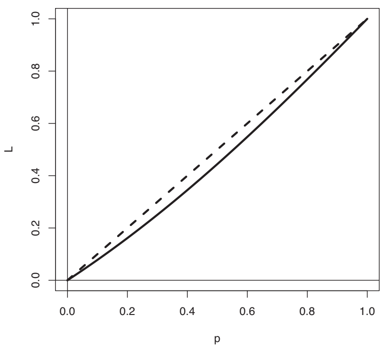

FIGURE 8. - $L(p)$ *curve of the probability density function of Figure 7*

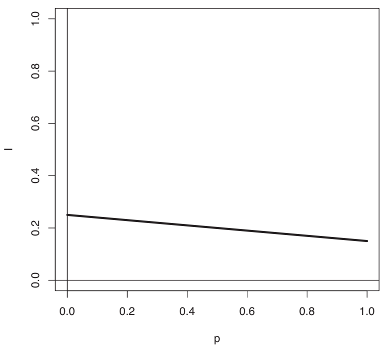

FIGURE 9. - *Linear* $I(p)$ *curve of the probability density function of Figure 7*

Running the above procedure for several values of p, and plotting the obtained values, the probability density function with linear $I(p)$ curve can be shown.

In Figures 4, 7 and 10, some density functions for different values of a and b are presented. For each density function, the corresponding Lorenz curve and $I(p)$ curve are provided.

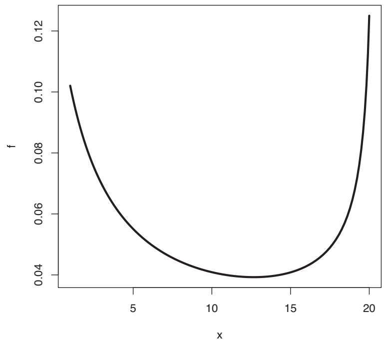

FIGURE 10. - *The probability density function with* $I(p) = -0,4p + 0,9$ *and* $\mu = 10$

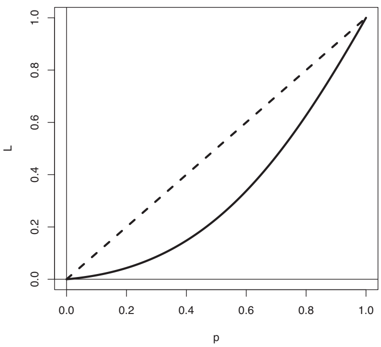

FIGURE 11. - $L(p)$ *curve of the probability density function of Figure 10*

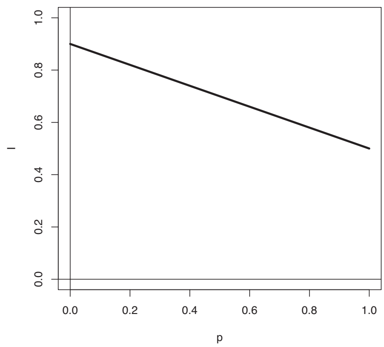

FIGURE 12. - *Linear* $I(p)$ *curve of the probability density function of Figure 10*

5. SOME CONSIDERATIONS REGARDING THE LINEAR CASE

In this section some features related to the linearity of $I(p)$ curve are analyzed.

The initial characteristic is that whenever a continuous random variable X has a linear $I(p)$ curve, the inequality index I of X can easily be calculated from the value of the corresponding $I(p)$ curve in $p = 0,5$. As Figure 13 shows, the two highlighted areas are equivalent, therefore I equals $I(0,5)$. By definition of index I, it means that

$$I = 1 - \frac{\overset{-}{M}_{(0,5)}}{\overset{+}{M}_{(0,5)}},$$

therefore, since it holds that (use equation (4) with $p = 0,5$):

$$2\mu = \overset{-}{M}_{(0,5)} + \overset{+}{M}_{(0,5)}$$

after a rearrangement, it follows that

$$I = \frac{2\mu - 2\bar{M}_{(0,5)}}{2\mu - \bar{M}_{(0,5)}}$$

and therefore:

$$I = \frac{1 - \frac{\bar{M}_{(0,5)}}{\mu}}{1 - \frac{\bar{M}_{(0,5)}}{2\mu}}.$$

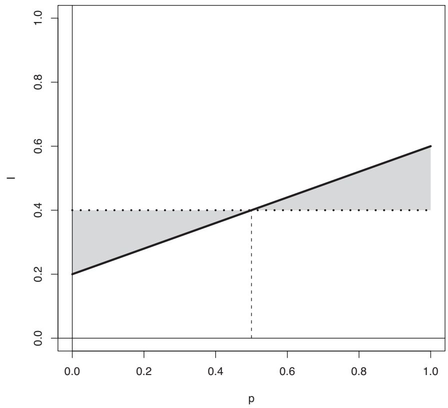

FIGURE 13. - *The inequality index I for linear $I(p)$ curve*

Another interesting issue is relating to the role of the two line parameters (a and b) from the inequality point of view. If (5) holds, a and b can be considered as parameters of the model with distribution function F. In particular, for a fixed b, the line parameter a is a direct inequality indicator for the $I(p)$ curve (the bigger a, the higher inequality) and therefore, by definition, this also holds true for the index I. On the other hand, for a fixed a, the line parameter b is a direct inequality indicator concerning the inequality $I(p)$ curve and the index I. It is important to note that the behavior of the $I(p)$ curve above considered, holds also for Lorenz curve $L(p)$, since the partial order based on the former is equivalent to the partial order based on the latter, as shown in Porro (2008). This ordering is inherited by Gini concentration ratio, whereby if one line parameter is fixed, the other one is likewise a direct inequality indicator for that inequality index.

6. TWO APPLICATIONS TO REAL DATA

The distribution model defined in the previous sections is not merely a theoretical exercise: there exist real situations in social sciences where it is possible to find a statistical variable generating an inequality $I(p)$ curve very similar to a straight line. The next two examples related to air traffic show evidence of this. The first example is about the number of passengers of the most important European airlines; the second one is about the most crowded worldwide airports.

In both applications, the empirical $I(p)$ ''curves'' are extended to the continuous case in order to compare the data results with the theoretical ones.

6.1 *Application 1*

As mentioned above, the first example is obtained following the analysis of the number of passengers for the 31 major European airlines in 2008. The data that come from AEA, Association of European Airlines (see AEA, 2009), are laid out in Table 1. From the data, the corresponding $I(p)$ curve is achieved and shown in Figure 14: the difference between this and a straight line is quite negligible.

To prove this, an usual linear regression has been performed, and the estimates of

TABLE 1. - *Number of passenger boarded by the major European airlines in 2008*

Airline	Airline Code	Passengers Boarded (Jan08-Dec08) x 1000
LUXAIR	LG	802.6
UKRAINE INTERNATIONAL AIRLINES	PS	920.1
ADRIA AIRWAYS	JP	1103.7
JAT AIRWAYS	JU	1157.8
ICELANDAIR	FI	1416.7
AIR MALTA	KM	1538.9
CYPRUS AIRWAYS	CY	1710.8
CROATIA AIRLINES	OU	1777.4
TAROM ROMANIAN AIR TRANSPORT	RO	1778.9
AEROSVIT	VV	1798.3
MALEV HUNGARIAN AIRLINES	MA	3122.9
LOT POLISH AIRLINES	LO	3966.7
CZECH AIRLINES	OK	4769.1
BRUSSELS AIRLINES	SN	5106.7
OLYMPIC AIRLINES	OA	5268.3
VIRGIN ATLANTIC AIRWAYS	VS	5685.4
FINNAIR	AY	6883.8
AIR ONE	AP	7404.1
TAP PORTUGAL	TP	8737.2
SPANAIR	JK	8793.7
AUSTRIAN	OS	9140.7
BMI	BD	9301.6
SWISS INTERNATIONAL AIRLINES	LX	13 319.8
ALITALIA	AZ	18 048.0
TURKISH AIRLINES	TK	21 870.4
IBERIA	IB	22 833.5
KLM ROYAL DUTCH AIRLINES	KL	23 844.1
SAS SCANDINAVIAN AIRLINES	SK	25 355.1
BRITISH AIRWAYS	BA	33 652.1
AIR FRANCE	AF	49 975.1
DEUTSCHE LUFTHANSA AG	LH	54 664.1

the line parameters have been obtained. The estimates are:

$$\hat{a} = -0.131 \qquad \textit{and} \qquad \hat{b} = 0.944. \tag{14}$$

The goodness of fit of the linear regression is very high, since $R^2 = 0.968$. A comparison between the theoretical distribution function implicitly provided by Theorem 1 with the parameters (14) and the empirical distribution function is shown in Figure 15.

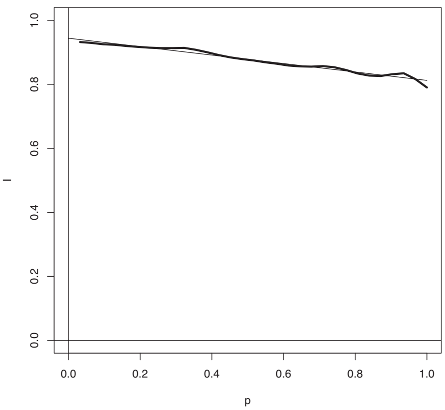

FIGURE 14. - *In bold, $I(p)$ curve for European airline dataset*

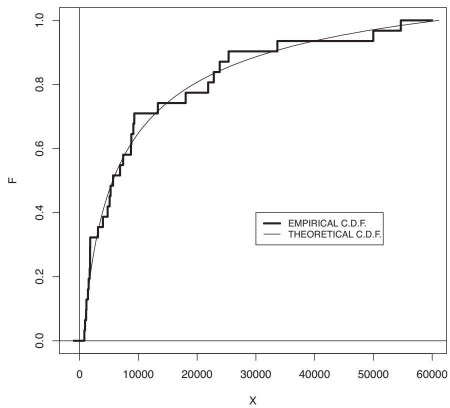

FIGURE 15. - *Theoretical and empirical c.d.f. for European airline dataset*

6.2 *Application 2*

The second example highlights the number of passengers for the 30 most crowded airports in the world. The data regarding 2008, that come from the Airport Council International (ACI) (see ACI, 2009), are reported in Table 2.

In Figure 16, the corresponding $I(p)$ curve is shown: as before, it is very similar to a straight line. It is worth noting that in this case the straight line approximating the $I(p)$ curve increases ($a > 0$), while in the previous application it decreases ($a < 0$).

The estimates of the line parameters by linear regression are

$$\hat{a} = 0.1653 \qquad \textit{and} \qquad \hat{b} = 0.2817. \tag{15}$$

In this case, the value of the goodness of fit is also very high, since R^2 equals 0.9201. In Figure 17, the empirical distribution function and the theoretical distribution function with the parameters provided in (15) are presented. It is interesting to note that for this dataset, if p lies in the interval $(0.6; 0.9)$ the line overestimates the inequality curve $I(p)$, whereas if p is greater, the line underestimates the inequality curve, as can be seen in Figure 16. This behavior can be also recognized trought the comparison of the distribution functions, in Figure 17: between 60 and 80 mln (passengers) the theoretical distribution function lies below the empirical one, but the reverse is true for higher values.

TABLE 2. - *Number of passengers for the major airports in 2008*

Location	Airport Code	Total Passengers 2008
Minneapolis, MN	MSP	34 056 443
Miami, FL	MIA	34 063 531
London (Gatwick)	LGW	34 214 740
Munich	MUC	34 530 593
Charlotte, NC	CLT	34 739 020
Rome	FCO	35 132 224
Detroit, MI	DTW	35 135 828
Newark, NJ	EWR	35 360 848
Orlando, FL	MCO	35 660 842
San Francisco, CA	SFO	37 234 592
Dubai	DXB	37 441 440
Singapore	SIN	37 694 824
Bangkok	BKK	38 603 490
Phoenix, AZ	PHX	39 891 193
Houston, TX	IAH	41 709 389
Las Vegas, NV	LAS	43 208 724
Amsterdam	AMS	47 430 019
New York, NY (JFK)	JFK	47 807 816
Hong Kong	HKG	47 857 846
Madrid	MAD	50 824 435
Denver, CO	DEN	51 245 334
Frankfurt	FRA	53 467 450
Beijing	PEK	55 937 289
Dallas, TX	DFW	57 093 187
Los Angeles, CA	LAX	59 497 539
Paris	CDG	60 874 681
Tokio	HND	66 754 829
London (Heathrow)	LHR	67 056 379
Chicago, IL	ORD	69 353 876
Atlanta, GA	ATL	90 039 280

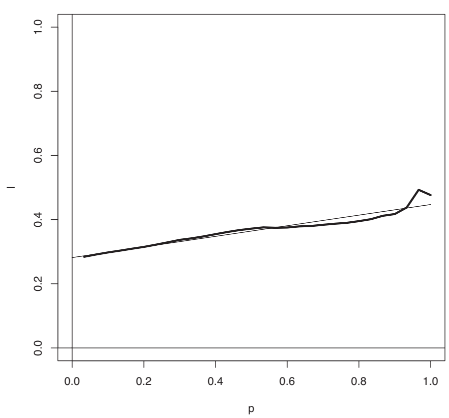

FIGURE 16. - *In bold, $I(p)$ curve for the airports dataset*

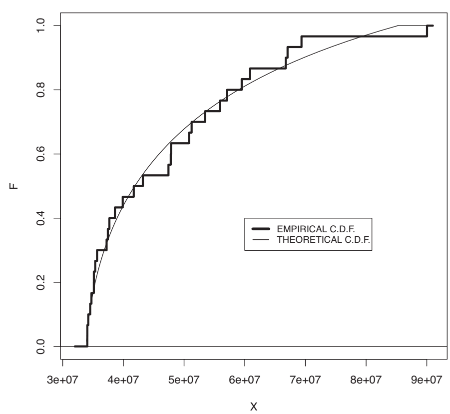

FIGURE 17. - *Theoretical and empirical c.d.f. for the airports dataset*

7. FINAL REMARKS

In this paper the distribution model with $I(p)$ curve given by $I(p) = ap + b$ is analyzed. This can be considered as the extension of the uniform case (where $a = 0$), previuosly considered in Polisicchio (2008). Unfortunately, the distribution function in the linear case can not be explicitly obtained for all the values of the line parameters a and b, but only for specific values. Nevertheless, following an ad-hoc procedure, it is possible to elaborate the graph of the probability density function of the random variable with linear inequality $I(p)$ curve. In order to identify the features of the two parameters of the line (a and b), a detailed analysis of their admissable values has been performed and their roles, in terms of inequality, have been determined.

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