Statistica & Applicazioni - Vol. IX, n. 2, 2011, pp. 129-158

POINT AND INTERVAL ESTIMATION FOR SOME FINANCIAL PERFORMANCE MEASURES

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SUMMARY

We study the estimators of three financial performance measures: the Sharpe Ratio, the Mean Difference Ratio and the Mean Absolute Deviation Ratio. The analysis is performed under two sets of assumptions. First, the case of i.i.d. Normal returns is considered. After that, relaxing the normality assumption, the case of i.i.d. returns is investigated. In both situations, we study the bias of the estimators and we propose their bias-corrected version. The exact and asymptotic distribution of the three estimators is derived under the assumption of i.i.d. Normal returns. Concerning the case of i.i.d. returns, the asymptotic distribution of the estimators is provided. The latter distributions are used to define exact or asymptotic confidence intervals for the three indices. Finally, we perform a simulation study in order to assess the efficiency of the bias corrected estimators, the coverage accuracy and the length of the asymptotic confidence intervals.

Keywords: *Financial Performance Measure, Sharpe Ratio, Mean Difference Ratio, Mean Absolute Deviation Ratio, Concentration Measures, Statistical Analysis of Financial Data.*

1. INTRODUCTION

Most of the known financial performance measures are defined as the ratio between a reward measure and a risk measure. For example, the well-known Sharpe Ratio, the Mean Difference Ratio (MD Ratio) and the Mean Absolute Deviation Ratio (MAD Ratio) have this structure. For all these three indices, the reward measure is the expected excess return (beyond some risk-free rate); while the risk measure differs among the indices. In particular, the risk is measured by the standard deviation of returns in the Sharpe Ratio, by the Gini mean difference in the MD Ratio and by the mean absolute deviation in the MAD Ratio.

The MD Ratio (see Shalit and Yitzhaki, 1984) and the MAD Ratio (see Konno and Yamazaki, 1991) were proposed to overcome the criticism against the use of the standard deviation as risk measure. These criticisms are motivated by the fact that the standard deviation is the natural risk measure only when the distribution of returns is Gaussian, a fact contradicted by empirical evidence which shows that the dis-

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Even if this paper results from the collaboration between the two authors, Sections 1, 2 and 8 have to be attributed to M. Zenga, while Sections 3, 4, 5, 6, and 7 to L. De Capitani.

tributions of financial returns are characterized by fatter tails than the Normal distribution and slight skewness.

In this paper we derive Confidence Intervals (CIs) for the three mentioned indices and we study the bias of their estimators. In detail, the paper is organized as follows. In Section 2 the definitions of Sharpe Ratio, MD Ratio and MAD Ratio are provided. In Section 3 we analyze the case of independent, identically normally-distributed returns (i.i.d. Normal returns) while in Section 4 we consider the case of independent and identically distributed returns (i.i.d. returns). In Section 5 we describe a simulation study performed in order to assess the coverage accuracy and the length of the asymptotic CIs and to study the bias of the estimators of the three indices. In Sections 6 and 7 we discuss the main results obtained in the simulations. Finally, section 8 is devoted to the conclusions.

In the following, we denote with F_H, μ_H and σ_H^2 the distribution function, the expectation and the variance of the random variable H, respectively. The Gini's mean difference of H is denoted by Δ_H and the mean absolute deviation of H is denoted by δ_H. Finally, σ_{HK} stands for the covariance between the random variables H and K.

2. THE SHARPE RATIO, THE MD RATIO AND THE MAD RATIO

Let X be the random variable describing the return of a risky financial asset and let Y be the random variable representing the return of the risk-free financial asset. Then, the random variable $D = X - Y$ describes the excess return of the risky financial asset with respect to the risk-free financial asset. The Sharpe Ratio (1966, 1994) is given by:

$$\psi^* = \frac{\mu_D}{\sqrt{\sigma_D^2}} = \frac{\mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}}}. \tag{1}$$

As pointed out in Sharpe (1994), this measure can be interpreted as the expected excess return per unit of risk where the risk is measured by the standard deviation of D.

Similarly, the MD Ratio and the MAD Ratio are, respectively, given by

$$\psi_\Delta^* = \frac{\mu_D}{\Delta_D} \qquad \text{and} \qquad \psi_\delta^* = \frac{\mu_D}{\delta_D}.$$

The interpretation of these two performance measures is similar to that of ψ^*.

Even if the formula (1) is the definition of the Sharpe Ratio proposed in Sharpe (1994), in literature the latter is usually defined as

$$\psi = \frac{(\mu_X - \mu_Y)}{\sigma_X}.$$

That because, theoretically, the risk-free financial asset has a constant return. Therefore, Y is degenerate on μ_Y and $\psi^* \equiv \psi$. According to most of the literature, here we adopt this ''simplified'' definition of Sharpe Ratio and we assume that the risk-free rate μ_Y is known.

A similar simplification can be made for the MD Ratio and the MAD Ratio. In particular, it is easy to verify that, for all $a \in \mathbb{R}$, $\Delta_{H+a} = \Delta_H$ and $\delta_{H+a} = \delta_H$. Then, if Y is degenerate on μ_Y, it follows that $\Delta_D = \Delta_X$ and $\delta_D = \delta_X$. Consequently, the simplified definitions of the MD Ratio and of the MAD Ratio are

$$\psi_\Delta = \frac{(\mu_X - \mu_Y)}{\Delta_X} \qquad \text{and} \qquad \psi_\delta = \frac{(\mu_X - \mu_Y)}{\delta_X}.$$

It can be noted that, under the assumption that μ_Y is a known constant, the just introduced simplified ratios depend only on the features of X. Then, to simplify the notation, we denote the risk-free rate by ξ and we drop the subscript X in σ_X^2, Δ_X, δ_X, and F_X, in the following.

3. THE CASE OF I.I.D. NORMAL RETURNS

Let X be a Normal random variable with parameters μ and σ: $X \sim N(\mu, \sigma^2)$. In this case the MD Ratio and the MAD Ratio can be expressed as a scale transformation of the Sharpe Ratio. In particular, it is well-known (see Johnson, Kotz and Balakrishnan, 1995a) that

$$\Delta = \frac{2\sigma}{\sqrt{\pi}} \qquad \text{and} \qquad \delta = \sigma\sqrt{\frac{2}{\pi}}. \tag{2}$$

Consequently

$$\psi_\Delta = \frac{\sqrt{\pi}}{2}\psi \qquad \text{and} \qquad \psi_\delta = \sqrt{\frac{\pi}{2}}\psi. \tag{3}$$

Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from the Normal distribution with parameters μ and σ^2. In order to estimate the Sharpe Ratio the first choice is to use the natural estimator

$$\widehat{\Psi} = \frac{(\bar{X} - \xi)}{\sqrt{S^2}},$$

where $\bar{X}$ and S^2 denote the sample mean and the unbiased sample variance, respectively:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i\,; \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

By the relations (3), the random variable

$$\widehat{\Psi}^*_\Delta = \frac{\sqrt{\pi}}{2}\widehat{\Psi}$$

is an estimator of the MD Ratio. Similarly, the random variable

$$\widehat{\Psi}^*_\delta = \sqrt{\frac{\pi}{2}}\widehat{\Psi}$$

is an estimator of the MAD Ratio.

The properties of $\widehat{\Psi}^*_\Delta$ and $\widehat{\Psi}^*_\delta$ can be easily deduced from those of $\widehat{\Psi}$. For this reason, in the remainder of this section, we will mainly focus on $\widehat{\Psi}$.

The asymptotic distribution of $\widehat{\Psi}$ was obtained by Jobson and Korkie (1981) and it is given by

$$\sqrt{n}(\widehat{\Psi} - \psi) \overset{a}{\sim} N\left(0; \frac{2 + \psi^2}{2}\right). \tag{4}$$

Here, we obtain the exact distribution of $\widehat{\Psi}$.

Proposition 1
Under the assumption of i.i.d. Normal returns, the random variable $\sqrt{n}\widehat{\Psi}$ follows the non-central t distribution with $(n-1)$ degrees of freedom and non-centrality parameter $\sqrt{n}\psi$.

Proof:
First, observe that

$$\sqrt{n}\widehat{\Psi} = \frac{\sqrt{n}\dfrac{\bar{X} - \mu}{\sigma} + \sqrt{n}\psi}{\sqrt{\left(\dfrac{(n-1)S^2}{\sigma^2}\right)/(n-1)}}. \tag{5}$$

From expression (5), it is clear that $\sqrt{n}\widehat{\Psi}$ is the ratio of two random variables. The numerator is Gaussian with expected value $\sqrt{n}\psi$ and standard deviation 1. The denominator is the square root of a chi square random variable divided by its degrees of freedom (df). Further, $\bar{X}$ and S^2 are independent since X is normally distributed (see Mood, Graybill and Boes, 1974, Theorem 8, p. 243). As a consequence, the numerator and the denominator of (5) are independent and the distribution of $\sqrt{n}\widehat{\Psi}$ is obtained from the definition of the non-central t distribution (see Johnson, Kotz and Balakrishnan, 1995b, ch. 31, p. 508).

Remark 1
It is worthwhile to note that the distribution of $\widehat{\Psi}$ depends on the parameter ψ and not on the particular values of μ and σ. In detail, let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$. Suppose that

$$\frac{\mu_1 - \xi}{\sigma_1} = \frac{\mu_2 - \xi}{\sigma_2} = \psi^* \quad .$$

In this case the distribution of the estimators $\widehat{\Psi}$ is non-central t with $(n-1)$ *df* and non-centrality parameter $\sqrt{n}\psi^*$ whether we sample from F_1 or F_2.

We use the result in Proposition 1 to define an exact CI fo ψ. In detail, let $t_{\nu,p}(a)$ be the p-quantile of a non-central t distribution with ν df and non-centrality parameter a. Let $\widehat{\psi}$ be an estimate of ψ. The extremes of the $(1-\alpha)$-CI for ψ, denoted by $(\psi_-; \psi_+)$, are the following (see Casella and Berger, 2002, ch. 9, p. 432; Johnson, Kotz and Balakrishnan, 1995b, ch. 31, p. 510):

$$\psi_+ : \quad t_{(n-1),\frac{\alpha}{2}}(\sqrt{n}\psi_+) = \sqrt{n} \cdot \widehat{\psi}$$
$$\psi_- : \quad t_{(n-1),1-\frac{\alpha}{2}}(\sqrt{n}\psi_-) = \sqrt{n} \cdot \widehat{\psi}.$$

The last two equations cannot be analytically solved but their solutions can be easily computed numerically.

The CI just introduced improves the one obtained by analytical inversion from the asymptotic distribution (4):

$$\left(\widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{2+\widehat{\Psi}^2}{2n}} \; ; \; \widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\frac{2+\widehat{\Psi}^2}{2n}}\right). \tag{6}$$

Another aspect to consider in the estimation of ψ is the bias of $\widehat{\Psi}$. In Miller and Gher (1978) it is shown that the estimator $\widehat{\Psi}$ is biased and $E(\widehat{\Psi}) = \psi \cdot d$, where

$$d = \sqrt{\frac{n-1}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

It is possible to prove that the bias factor d is greater than 1 for all $n > 2$. Then, the estimator $\widehat{\Psi}$ tends to overestimate (underestimate) ψ when the later is positive (negative). Further, the estimator $\widehat{\Psi}_u = d^{-1}\widehat{\Psi}$ is unbiased and it is more efficient than $\widehat{\Psi}$.

In Jobson and Korkie (1981), the following, easy to calculate, approximation of the bias factor d is given[1]:

$$d \approx d_1 = \left(1 + \frac{3}{4(n-1)} + \frac{25}{32(n-1)^2}\right). \tag{7}$$

Here, we provide a further and more accurate approximation of d. To do this, first note that

$$d = \frac{\sqrt{2(n-1)}}{(n-2)} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right). \tag{8}$$

1 It is worthwhile to note that the approximation of the bias factor d, given in the original article of Jobson and Korkie (1981), contains a typo. In particular the approximation given in Jobson and Korkie (1981) can be obtained from expression (7) replacing $(n-1)$ with n: $d \approx (1 + 3/(4n) + 25/(32n^2))$. We also point out that this approximation is sometimes recalled in the just cited incorrect version (see, for example, Knight and Satchell, 2005). Finally, we remark that the incorrect approximation is less accurate than the correct one.

In Graham, Knuth, and Patashnik (1994) (see response to problems 9.60), it is shown that

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \sqrt{\frac{n-1}{2}}\left(1 - \frac{1}{4(n-1)} + \frac{1}{32(n-1)^2} + O(n^{-3})\right) \tag{9}$$

and, from expressions (8) and (9), we obtain that

$$d \approx d_2 = \frac{n-1}{n-2}\left(1 - \frac{1}{4(n-1)} + \frac{1}{32(n-1)^2}\right) . \tag{10}$$

It is easy to check, by direct computation, that the approximation (10) is more accurate than approximation (7). Moreover, both d_1 and d_2 are greater than 1 for all $n > 2$ and, in more detail, $1 < d_1 < d_2 < d$ for all $n > 2$. As a consequence, the approximately unbiased estimators $\widehat{\Psi}_{u1} = d_1^{-1}\widehat{\Psi}$ and $\widehat{\Psi}_{u2} = d_2^{-1}\widehat{\Psi}$ are more efficient than $\widehat{\Psi}$. Naturally, $\widehat{\Psi}_u$ is more efficient than $\widehat{\Psi}_{u2}$ which is more efficient than $\widehat{\Psi}_{u1}$.

As aforementioned, all the results related to the estimator $\widehat{\Psi}$ can be turned into results concerning $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\Delta$. In particular, we have that $\sqrt{n}\widehat{\Psi} \overset{d}{=} \sqrt{\frac{4n}{\pi}}\widehat{\Psi}^*_\Delta \overset{d}{=} \sqrt{\frac{2n}{\pi}}\widehat{\Psi}^*_\delta$ and an exact CI for ψ_Δ and ψ_δ can be obtained by multiplying the extremes of the CI for ψ by $\frac{\sqrt{\pi}}{2}$ and by $\sqrt{\frac{\pi}{2}}$, respectively. In detail, the exact CI for ψ_δ is

$$\left(\sqrt{\frac{\pi}{2}}\psi_-; \sqrt{\frac{\pi}{2}}\psi_+\right),$$

while the exact CI for ψ_Δ is

$$\left(\frac{\sqrt{\pi}}{2}\psi_-; \frac{\sqrt{\pi}}{2}\psi_+\right).$$

Analogously, asymptotic CIs for ψ_δ and ψ_Δ, based on estimators $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\Delta$, can be obtained from the CI (6):

$$\left(\widehat{\Psi}^*_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + \widehat{\Psi}^{*2}_\delta}{2n}}\ ;\ \widehat{\Psi}^*_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + \widehat{\Psi}^{*2}_\delta}{2n}}\right), \tag{11}$$

$$\left(\widehat{\Psi}^*_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{\widehat{\Psi}^{*2}_\Delta}{2n}}\ ;\ \widehat{\Psi}^*_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{\widehat{\Psi}^{*2}_\Delta}{2n}}\right). \tag{12}$$

Furthermore, the estimators $\widehat{\Psi}^*_{\Delta u} = d^{-1}\widehat{\Psi}^*_\Delta$ and $\widehat{\Psi}^*_{\delta u} = d^{-1}\widehat{\Psi}^*_\delta$ are unbiased and more efficient than $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\delta$, respectively. As for the Sharpe Ratio, it is possible to introduce approximately unbiased estimators for the MD Ratio and the MAD Ratio using the approximations of the bias factor (7) and (10). Naturally, the usefulness of the approximations d_1 and d_2 is low since, nowadays, computers manage to calculate the values of the Γ function even for quite large values of its argument.

Also for the MD Ratio and the MAD Ratio we can observe that the distribution of $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\Delta$ depends only on the true value of ψ_δ and ψ_Δ, not on the single values of μ, Δ and δ (see Remark 1).

In Section 4 we will show some additional asymptotic results concerning the MD Ratio and the MAD Ratio under the assumption of i.i.d. Normal returns. These results will be obtained as a special case of those derived under the assumption of i.i.d. returns.

4. THE CASE OF I.I.D. RETURNS

The estimators for the MD Ratio and the MAD Ratio, proposed in the previous section, stem from the particular relations existing among the standard deviation, the Gini mean difference and the mean absolute deviation of a Normal random variable. In this section, we do not specify a particular parametric model for F. Consequently, the estimators of ψ_Δ and ψ_δ cannot be defined starting from the estimators of the parameters of F. Then, we consider the following natural estimators for ψ_δ and ψ_Δ:

$$\widehat{\Psi}_\Delta = \frac{\bar{X} - \xi}{\hat{\Delta}} \qquad \text{and} \qquad \widehat{\Psi}_\delta = \frac{\bar{X} - \xi}{\hat{\delta}}$$

where

$$\hat{\Delta} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_i - X_j| \qquad \text{and} \qquad \hat{\delta} = \frac{1}{n} \sum_{i=1}^{n} |X_i - \bar{X}|.$$

The exact distributions of $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ cannot be derived because F is unknown. However, in the following, we will obtain their asymptotic distribution. For that purpose, it is useful to note that the MD Ratio and the MAD Ratio are strictly related to two well-known concentration measures. In more detail, $\psi_\Delta = (2G)^{-1}$ and $\psi_\delta = (2P)^{-1}$ where $G = \Delta/(2\mu)$ and $P = \delta/(2\mu)$ are the Gini concentration ratio and the Pietra concentration ratio of the random variable $X - \xi$. Consequently, we will use several results obtained for the estimators of G and P in order to study the features of $\widehat{\Psi}_\Delta$ and $\widehat{\Psi}_\delta$.

4.1 *Asymptotic distribution of $\widehat{\Psi}$ and CI for ψ*

The asymptotic distribution of $\widehat{\Psi}$ is well known and it was derived by Lo (2002). However, here, we briefly recall his results for the sake of completeness.

The fundamental, well known result, necessary in order to obtain the asymptotic distribution of $\widehat{\Psi}$, is recalled in the following theorem (see Serfling, 1980, p. 114).

Theorem 1
Let $X_1, \ldots, X_n$ be an i.i.d. sample from F ad assume that $E[X_1^4] < \infty$. It follows that

$$\sqrt{n}\begin{bmatrix} \bar{X} - \mu \\ S^2 - \sigma^2 \end{bmatrix} \xrightarrow{d} \mathrm{NB}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{bmatrix}\right) \tag{13}$$

where $\mu_k = E[(X - \mu)^k]$ and NB *means ''bivariate Normal''.*

Starting from expression (13), Lo (2002) derived the asymptotic distribution of $\widehat{\Psi}$ by the delta-method (see Serfling, 1980, Theorem A in Section 3.3, p. 122). He obtained:

$$\sqrt{n}(\widehat{\Psi} - \psi) \xrightarrow{d} N(0; V) \quad \text{where} \quad V = 1 - \frac{\mu_3}{\sigma^3}\psi + \left(\frac{\mu_4}{\sigma^4} - 1\right)\frac{\psi^2}{4}.$$

The variance V can be consistently estimated by

$$\widehat{V} = 1 - \frac{\hat{\mu}_3}{S^3}\widehat{\Psi} + \left(\frac{\hat{\mu}_4}{S^4} - 1\right)\frac{\widehat{\Psi}^2}{4}$$

where $\hat{\mu}_3$ and $\hat{\mu}_4$ denote the third and the fourth sample central moment:

$$\hat{\mu}_3 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^3; \qquad \hat{\mu}_4 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4.$$

Then, the asymptotic $(1 - \alpha)$-CI for ψ is

$$\left(\widehat{\Psi} - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}/n}\,;\ \widehat{\Psi} + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}/n}\right). \tag{14}$$

4.2 *Asymptotic distribution of $\widehat{\Psi}_\Delta$ and CI for ψ_Δ*

In order to obtain the asymptotic distribution of $\widehat{\Psi}_\Delta$, we follow a procedure quite similar to that followed by Lo (2002). In detail, we recall the following fundamental result, due to Hoeffding (1948).

Theorem 2
Let $X_1, \ldots, X_n$ be an i.i.d. sample from F ad assume that $E[X_1^2] < \infty$. Then

$$\sqrt{n}\begin{bmatrix} \bar{X} - \mu \\ \hat{\Delta} - \Delta \end{bmatrix} \xrightarrow{d} \mathrm{NB}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}; \begin{bmatrix} \sigma^2 & \gamma \\ \gamma & \zeta^2 \end{bmatrix}\right) \tag{15}$$

where $\gamma = 2(\mathcal{D} - \mu\Delta)$, $\zeta^2 = 4(\mathcal{F} - \Delta^2)$,

$$\mathcal{D} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x|x-y|dF(y)dF(x), \quad \textit{and}$$

$$\mathcal{F} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y||x-z|dF(z)dF(y)dF(x).$$

In Hoeffding (1948) this result is used to derive the asymptotic distribution of the Gini Concentration Ratio. Here, starting from (15), and applying the delta-method, we obtain the following result.

Proposition 2
Under the assumptions of Theorem 2, we have that:

$$\sqrt{n}(\widehat{\Psi}_\Delta - \psi_\Delta) \xrightarrow{d} \mathcal{N}(0; V_\Delta)\ , \quad \text{where} \quad V_\Delta = \frac{\sigma^2}{\Delta^2} - 2\frac{\gamma}{\Delta^2}\psi_\Delta + \frac{\zeta^2}{\Delta^2}\psi_\Delta^2.$$

An unbiased and consistent estimator for ζ^2 is given by (see Zenga, 2004):

$$\hat{\zeta}^2 = \frac{4n}{(n-2)(n-3)}\left[S^2 + (n-2)\hat{\mathcal{F}} - \frac{(2n-3)}{2}\hat{\Delta}^2\right]$$

where

$$\hat{\mathcal{F}} = \frac{1}{n(n-1)(n-2)}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}|X_i - X_j||X_i - X_l| - \frac{2S^2}{n-2}.$$

An unbiased and consistent estimator for γ is given by (see Polisicchio, 1997):

$$\hat{\gamma} = \frac{2n}{n-2}[\hat{\mathcal{D}} - \bar{X}\hat{\Delta}] \qquad \text{where} \qquad \hat{\mathcal{D}} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n} X_i|X_i - X_j|.$$

So, the variance V_Δ can be consistently estimated by

$$\widehat{V}_\Delta = \frac{S^2}{\hat{\Delta}^2} - 2\frac{\hat{\gamma}}{\hat{\Delta}^2}\psi_\Delta + \frac{\hat{\zeta}^2}{\hat{\Delta}^2}\widehat{\Psi}_\Delta^2$$

and we introduce the following asymptotic $(1-\alpha)$-CI for ψ_Δ:

$$\left(\widehat{\Psi}_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\Delta/n}\, ;\, \widehat{\Psi}_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\Delta/n}\right)\ . \tag{16}$$

Remark 2
Under the additional assumption that X is normally distributed, the expression of the variance V_Δ can be simplified. In detail, in Polisicchio and Zini (2000) it is shown that the covariance between the sample mean and the sample Gini mean dif-

ference is null when sampling from a symmetric distribution. Further, in Zenga, Polisicchio, and Greselin (2004) the expression of the functional $\mathcal{F}$ for the Normal distribution is given:

$$\mathcal{F} = \frac{\sigma^2}{3\pi}(\pi + 6\sqrt{3}). \tag{17}$$

From expression (17) and remembering relation (2), the variance of V_Δ results:

$$V_\Delta = \frac{\pi}{4} + \left(\frac{\pi + 6\sqrt{3} - 12}{3}\right)\psi_\Delta^2.$$

Consequently, under the assumption of i.i.d. Normal returns, we provide the following asymptotic $(1-\alpha)$-CI for ψ_Δ based on estimator $\widehat{\Psi}_\Delta$, which can be used in place of that based on $\widehat{\Psi}^*_\Delta$, introduced in Section 3:

$$\left(\widehat{\Psi}_\Delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{c}{3n}\widehat{\Psi}^2_\Delta}\ ;\ \widehat{\Psi}_\Delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi}{4n} + \frac{c}{3n}\widehat{\Psi}^2_\Delta}\right), \tag{18}$$

where $c = \pi + 6\sqrt{3} - 12$.

4.3 *Asymptotic distribution of $\widehat{\Psi}_\delta$ and CI for ψ_δ*

We derive the asymptotic distribution of $\widehat{\Psi}_\delta$, starting from the following result, due to Gastwirth (1974).

Theorem 3
Let $X_1, ..., X_n$ be an i.i.d. sample from F ad assume that $E[X_1^2] < \infty$. Then

$$\sqrt{n}\begin{bmatrix}\bar{X} - \mu \\ \hat{\delta} - \delta\end{bmatrix} \xrightarrow{d} \mathrm{NB}\left(\begin{bmatrix}0 \\ 0\end{bmatrix};\ \begin{bmatrix}\sigma^2 & \kappa \\ \kappa & \upsilon^2\end{bmatrix}\right) \tag{19}$$

where

$$\upsilon^2 = 4p^2\sigma^2 + 4(1-2p)\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) - \delta^2\ ,$$

$$\kappa = 2p\sigma^2 - 2\int_{-\infty}^{\mu}(x-\mu)^2 dF(x)\ , \qquad \text{and} \qquad p = F(\mu).$$

Gastwirth (1974) used this result to obtain the asymptotic distribution of the Pietra concentration ratio. Here, starting from (19), and applying the delta-method, we obtain the following result.

Proposition 3
Under the assumptions of Theorem 3, we have that:

$$\sqrt{n}(\widehat{\Psi}_\delta - \delta) \xrightarrow{d} \mathcal{N}(0; V_\delta)\ , \quad \text{where} \quad V_\delta = \frac{\sigma^2}{\delta^2} - 2\frac{\kappa}{\delta^2}\psi_\delta + \frac{\upsilon^2}{\delta^2}\psi_\delta^2.$$

A consistent estimator of υ^2 is given by $\hat{\upsilon}^2 = (4\hat{p}^2 S^2 + 4(1-2\hat{p})\hat{\mu}_2^- - \hat{\delta}^2)$ where

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} I_i\,, \quad \hat{\mu}_2^- = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 I_i \quad \text{and} \quad I_i = \begin{cases} 1\,, & X_i < \bar{X} \\ 0\,, & \text{otherwise.} \end{cases}$$

A consistent estimator of κ is given by $\hat{\kappa} = (2\hat{p}S^2 - 2\hat{\mu}_2^-)$.
Consequently,

$$\widehat{V}_\delta = \frac{S^2}{\hat{\delta}^2} - 2\frac{\hat{\kappa}}{\hat{\delta}^2}\widehat{\Psi}_\delta + \frac{\hat{\upsilon}^2}{\hat{\delta}^2}\widehat{\Psi}_\delta^2$$

is consistent for V_δ and

$$\left(\widehat{\Psi}_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\delta/n}\,;\ \widehat{\Psi}_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{V}_\delta/n}\right) \tag{20}$$

is an asymptotic $(1-\alpha)$-CI for ψ_δ.

Remark 3

Under the further assumption of symmetry of F, it follows that $p = 1/2$ and $\int_{-\infty}^{\mu}(x-\mu)^2 dF(x) = \sigma^2/2$. Then $\kappa = 0$, $\upsilon^2 = \sigma^2 - \delta^2$, and $V_\delta = \sigma^2/\delta^2 + [(\sigma^2 - \delta^2)/\delta^2]\psi_\delta^2$. Moreover, if F is Gaussian, from expressions (2) we obtain that $V_\delta = \pi/2 + [(\pi-2)/2]\psi_\delta^2$. As a consequence, under the assumption of i.i.d. Normal returns we introduce the following asymptotic $(1-\alpha)$-CI based on estimator $\widehat{\Psi}_\delta$, which can be used in place of that based on $\widehat{\Psi}_\delta^*$, introduced in Section 3:

$$\left(\widehat{\Psi}_\delta - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + (\pi-2)\widehat{\Psi}_\delta^2}{2n}}\,;\ \widehat{\Psi}_\delta + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\pi + (\pi-2)\widehat{\Psi}_\delta^2}{2n}}\right). \tag{21}$$

Remark 4

Note that it is necessary to assume the existence of the fourth moment of X in order to derive the asymptotic distribution of $\widehat{\Psi}$ while it is only necessary to assume the existence of the second moment of X to obtain the asymptotic distributions of $\widehat{\Psi}_\Delta$ and $\widehat{\Psi}_\delta$. This observation is important since empirical evidence suggests that the distribution of the returns of many financial activities may have infinite fourth moment (especially when high frequency returns are considered as shown in Gençay, Dacorogna, Muller, Picket and Olsen, 2001). In these cases the CI for ψ provided in expression (14) cannot be used. This is a potential limitation of the Sharpe Ratio.

Remark 5

An observation analogous to that of Remark 1 can be made also in the context of the i.i.d. returns. In detail, if two financial activities have the same Sharpe Ratio (MD Ratio or MAD Ratio) and the distributions of their returns belong to the same location-scale family, then the asymptotic distribution of the estimators of the two Sharpe Ratios (MD Ratios or MAD Ratios) is the same. To be more clear, let X_1 and X_2 be the random variables describing the returns of the two financial activities and denote all the objects relating to X_i by the subscript i (e.g., ψ_i is the Sharpe Ratio of X_i, $\mathcal{F}_i$ is the functional $\mathcal{F}$ associated to the distribution of X_i, and so on). Let $X_2 \stackrel{d}{=} kX_1 + h$ so that the distributions F_1 and F_2 belong to the same location-scale family. It can be easily shown that, if $h = (1-k)\xi$ then $\psi_1 = \psi_2$ and $V_1 = V_2$. Then, asymptotically, $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ have the same distribution. Analogously, it can be shown that $\mathcal{D}_2 = k^2(\mathcal{D}_1 - \mu_1\Delta_1)$ and $\mathcal{F}_2 = k^2\mathcal{F}_1$. As a consequence, if $h = (1-k)\xi$ then $\psi_{\Delta 1} = \psi_{\Delta 2}$ and $V_{\Delta 1} = V_{\Delta 2}$ so that $\widehat{\Psi}_{\Delta 1}$ and $\widehat{\Psi}_{\Delta 2}$ have the same asymptotic distribution. A similar result can be obtained also for the MAD Ratio since $\upsilon_2^2 = k^2\upsilon_1^2$ and $\kappa_2 = k^2\kappa_1$.

4.4 *Bias of the estimators $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$*

The estimators $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ are biased. Unlike the case of i.i.d. Normal returns, in this context is not possible to determine exactly their bias since F is not specified. Then, we approximate the expectation of $\widehat{\Psi}$, $\widehat{\Psi}_\Delta$, and $\widehat{\Psi}_\delta$ recalling the following Taylor series approximation of the function $f(x,y) = x/y$ around the point (a,b):

$$\frac{x}{y} \approx \frac{a}{b} + \frac{1}{b}(x-a) + \frac{a}{b^2}(y-b) - \frac{1}{b^2}(x-a)(y-b) + \frac{a}{b^3}(y-b)^2. \tag{22}$$

Concerning the case of the estimator $\widehat{\Psi}$, let $(a,b) = (\mu - \xi, \sigma)$ and let $\bar{X}$ and S replace x and y, respectively, in expression (22). Taking expectation on both sides of the resulting formula, we obtain an approximation of $E[\widehat{\Psi}]$:

$$\begin{aligned} E\left[\frac{\bar{X}-\xi}{S}\right] &\approx E\left[\psi + \frac{\bar{X}-\mu}{\sigma} + \psi\left(\frac{S-\sigma}{\sigma}\right) - \frac{(\bar{X}-\mu)(S-\sigma)}{\sigma^2} + \psi\left(\frac{S-\sigma}{\sigma}\right)^2\right] \\ &= \psi\left(2 - \frac{E[S]}{\sigma}\right) - \frac{E[(\bar{X}-\mu)(S-\sigma)]}{\sigma^2}. \end{aligned} \tag{23}$$

The above expression depends on the unknown expectations $E[S]$ and $E[(\bar{X}-\mu)(S-\sigma)]$ which can be approximated using, once again, a Taylor series expansion. In detail, an approximation of $E[S]$ can be derived starting from the following Taylor approximation:

$$\sqrt{y} \approx \sqrt{b} + \frac{1}{2\sqrt{b}}(y-b) - \frac{1}{8\sqrt{b^3}}(y-b)^2. \tag{24}$$

Let $b = \sigma^2$ and let S^2 replace y in (24). Taking expectation on both side of the resulting expression and remembering that (see Johnson, 1995b)

$$E[(S^2-\sigma^2)^2] = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right),$$

we obtain

$$E[S] \approx \sigma - \frac{1}{8\sigma^3} E[(S^2 - \sigma^2)^2] = \sigma \left[1 - \frac{1}{8n} \left(\frac{\mu_4}{\sigma^4} - 3 \right) - \frac{1}{4(n-1)} \right]. \tag{25}$$

In order to approximate the value of $E[(\bar{X} - \mu)(S - \sigma)]$, we first observe that

$$E[(\bar{X} - \mu)(S - \sigma)] = E[\bar{X}S] - \mu E[S] = \sigma_{\bar{X}S}.$$

Further, we recall the following Taylor series approximation:

$$x\sqrt{y} \approx a\sqrt{b} + \sqrt{b}(x-a) + \frac{a}{2\sqrt{b}}(y-b) + \frac{1}{2\sqrt{b}}(x-a)(y-b) - \frac{a}{8\sqrt{b^3}}(y-b)^2. \tag{26}$$

Let $(a, b) = (\mu, \sigma^2)$ and let $\bar{X}$ and S^2 replace x and y, respectively, in (26). After taking expectation on both sides, the following expression is obtained:

$$E[\bar{X}S] \approx \frac{\mu_3}{2n\sigma} + \mu \left(\sigma - \frac{1}{8\sigma^3} E[(S^2 - \sigma^2)^2] \right).$$

Consequently

$$\sigma_{\bar{X}S} \approx \frac{\mu_3}{2n\sigma}. \tag{27}$$

Finally, plugging (25) and (27) into (23), we obtain

$$E[\widehat{\Psi}] \approx \psi \left[1 + \frac{1}{4(n-1)} + \frac{1}{8n} \left(\frac{\mu_4}{\sigma^4} - 3 \right) \right] - \frac{1}{2n} \frac{\mu_3}{\sigma^3}. \tag{28}$$

In a similar way, the following approximations are derived:

$$E[\widehat{\Psi}_\Delta] \approx \psi_\Delta \left(1 + \frac{\zeta^2}{n\Delta^2} \right) - \frac{\gamma}{n\Delta^2} \;; \qquad E[\widehat{\Psi}_\delta] \approx \psi_\delta \left(1 + \frac{\upsilon^2}{n\delta^2} \right) - \frac{\kappa}{n\delta^2}. \tag{29}$$

From (28) and (29) it turns out that the estimators

$$\widehat{\Psi}'_u = \left(\widehat{\Psi} + \frac{1}{2n} \frac{\hat{\mu}_3}{S^3} \right) \left[1 + \frac{1}{4(n-1)} + \frac{1}{8n} \left(\frac{\hat{\mu}_4}{S^4} - 3 \right) \right]^{-1}, \tag{30}$$

$$\widehat{\Psi}_{\Delta u} = \left(\widehat{\Psi}_\Delta + \frac{\hat{\gamma}}{n\hat{\Delta}^2} \right) \left(1 + \frac{\hat{\zeta}^2}{n\hat{\Delta}^2} \right)^{-1}, \text{ and } \widehat{\Psi}_{\delta u} = \left(\widehat{\Psi}_\delta + \frac{\hat{\kappa}}{n\hat{\delta}^2} \right) \left(1 + \frac{\hat{\upsilon}^2}{n\hat{\delta}^2} \right)^{-1} \tag{31}$$

are approximatively unbiased for ψ, ψ_Δ, and ψ_δ, respectively. However, the variability of these bias-corrected estimators could be much higher than the variability of uncorrected ones, because of the variability of the estimators S, $\hat{\mu}_3$, $\hat{\mu}_4$, $\hat{\gamma}$, $\hat{\Delta}$, $\hat{\zeta}^2$, $\hat{\kappa}$, $\hat{\delta}^2$, and $\hat{\upsilon}^2$.

5. DESIGN OF THE SIMULATION STUDY

In order to assess the coverage accuracy and the length of the asymptotic CIs and to compare the efficiency of the approximatively unbiased estimators with that of the natural ones, we perform a wide simulation study. Several scenarios are considered both concerning the case of i.i.d. Normal returns and the case of i.i.d. returns. In each scenario, the coverage accuracy of the asymptotic CIs is evaluated estimating the actual coverage of the CI by the proportion of simulated CIs containing the true value of the ratio. Further, the simulated Average Length (AL) of the different CIs is computed. Finally, in order to compare the efficiency of the approximately bias-corrected estimators and the uncorrected ones, we calculate in each scenario the simulated MSE and the simulated bias of all the estimators.

The differences among the scenarios considered concerns: the distribution of the random variable X (12 different distributions), the standard deviation σ of X (3 different values), the value of ψ (3 different values), the sample size (4 different values), the nominal coverage (4 different values). Globally, $(12 \times 3 \times 3 \times 4 \times 4) = 1728$ scenarios are investigated.

In detail, for each of the 12 distributions considered, we analyze the following scenarios:

- sample sizes: 50, 100, 200, 400;
- number of replications: 10^4;
- nominal coverages of the asymptotic CIs: 90%, 95%, 97.5%, 99%;
- value of the (daily) risk-free rate: 0.000068 (which corresponds to an annual rate of return of about 2.5%);
- values of the standard deviation of X: $\sigma = 0.01$, $\sigma = 0.05$, $\sigma = 0.1$. The different values of σ are chosen coherently with the values of the standard deviation of daily returns for the equities in the S&P 100 during the period 2005-2007;
- values of the Sharpe Ratio: $\psi_1 = 0.05$, $\psi_2 = 0.25$, $\psi_3 = 0.5$. In practice, for each value of σ, three different value of μ are considered. The first value of μ gives rise to a Sharpe Ratio of 0.05, the second gives rise to $\psi = 0.25$, finally, the third gives rise to $\psi = 0.5$. As for the values of σ, the different values of ψ are chosen on the base of the daily Sharpe Ratios of the equities in the S&P 100, calculated from the returns of period 2005-2007. The values of ψ_δ and ψ_Δ change among the scenarios coherently with the value of (μ, σ) and the distribution shape.

Under the hypotheses of i.i.d. Normal returns we evaluate the properties of the asymptotic CIs introduced in Section 3 and of the CIs (18) and (21), sampling from a Normal distribution. Under the assumption of i.i.d. returns we investigate the features of the CIs (14), (16), and (20) and the performances of the estimators (30) and (31) sampling from the following distributions: Normal (**N**), Laplace (**L**), Student's t with 5 ($\mathbf{t}_5$) and 3 ($\mathbf{t}_3$) df, Skew Normal with low and high degree of positive and negative skewness (**SNL**$\pm$, **SNH**$\pm$), Skew t with 5 df and with low and high degree of positive and negative skewness (**StL**$\pm$, **StH**$\pm$). The Normal distribution is taken into account also under the setting of i.i.d. returns because it is a useful cornerstone for the comment on the results obtained sampling from the other distributions. The Laplace distribution and the Student's t with 5 df are taken into account because they have fatter tails than the Normal. In detail, the Laplace distribution has fatter tails than the Normal, but it possess all the moments. On the contrary, the Student's t distribution with 5 df does not possess moments of order higher than 4. The Student's t with 3 df is investigated in order to evaluate the impact of the non-existence of the fourth moments on the properties of the asymptotic CI for the Sharpe Ratio (see Remark 4). We chose the Skew Normal distribution (see Azzalini and Capitanio, 1999) in order to evaluate the impact of the skewness. Four different parameter settings of the Skew Normal distribution are taken into consideration: the first (third) setting is characterized by a low degree of positive (negative) skewness (shape parameter equal to ± 2), while the second (fourth) is characterized by a higher degree of positive (negative) skewness (shape parameter equal to ± 3). Finally we consider the Skew t distribution with 5 df (see Azzalini and Capitanio, 2003) because it shows, at the same time, fat tails (only the moments up to order 4 exist) and skewness. As for the Skew Normal distribution, we investigate four different parameter settings: the first (third) is associated with a low degree of positive (negative) skewness (shape parameter equal to ± 0.5), the second (fourth) with a higher degree of positive (negative) skewness (shape parameter equal to ± 1). In Table 1 we give the values of the third standardized moment γ_1 (which is usually interpreted as an index of skewness) and the excess kurtosis γ_2 (the fourth standardized moment minus 3) associated with each of the distributions considered.

In the following we do not give all the detailed results obtained in the simulation study. Only the more interesting results are given and discussed. However, the detailed results are reported in De Capitani (2011). In Section 6 we discuss the results obtained under the assumption of i.i.d. Normal returns, and in Section 7 those obtained under the i.i.d. assumption.

6. RESULTS: I.I.D. NORMAL RETURNS

First, we analyze the coverage accuracy of the asymptotic CIs (6), (11), (12), (18) and (21). As explained in Remark 1, the features of the estimators $\widehat{\Psi}$, $\widehat{\Psi}^*_\delta$, $\widehat{\Psi}^*_\Delta$ do not depend on the value of σ but only on the sample size n and on the true values of ψ, ψ_δ, and ψ_Δ, respectively. Simulations show that an analogous result holds also

TABLE 1. - *Values of γ_1 and γ_2 for the 12 distributions considered. The values relating the Skew t distribution are obtained by numerical integration*

Distribution	γ_1	γ_2	Distribution	γ_1	γ_2
Normal	0	0	Laplace	0	3
t with 3 df	0	∞	t with 5 df	0	6
Skew N. (High as.)	± 0.6670	0.5098	Skew N. (Low as.)	± 0.4538	0.3051
Skew t (High as.)	± 1.0758	8.9208	Skew t (Low as.)	± 0.5527	6.8020

for the estimators $\widehat{\Psi}_\delta$ and $\widehat{\Psi}_\Delta$. Then, in Table 2 we give the averages over the different values of σ of the simulated coverages. The simulated coverages of the exact CIs are also given in order to evaluate the amount of variability due to the simulation procedure. The main result is that the actual coverage of all the asymptotic CIs is quite similar to the simulated coverage of the exact ones in all the scenarios considered. Then, a relatively small sample size of 50 is sufficient in order to assure a good coverage accuracy of the CIs (6), (11), (12), (18) and (21).

Concerning the length of the CIs, the simulated average lengths are given in Table 3. Even in this case, the value of σ does not significantly influence the results (as suggested by Remark 1) and, consequently, only the averages over the different va-

TABLE 2. - *Simulated coverage of the exact and asymptotic CIs for the Sharpe Ratio, MAD Ratio and MD Ratio under the assumption of i.i.d. Normal returns*

Exact CIs for ψ, ψ_δ, and ψ_Δ												
Ratio's value	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	89.99	94.97	97.49	99.00	89.99	95.03	97.53	99.02	90.01	95.00	97.49	99.03
100	89.93	94.97	97.55	99.06	89.89	94.96	97.55	99.03	89.79	94.95	97.52	99.05
200	89.91	95.01	97.56	98.98	89.87	95.04	97.55	99.00	89.80	95.05	97.57	99.01
400	89.88	95.05	97.59	99.04	89.93	95.05	97.56	99.02	89.96	95.05	97.51	99.05
Asymptotic CI for ψ and asymptotic CIs for ψ_δ and ψ_Δ based on $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\Delta$												
Ratio's value	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	89.82	94.84	97.42	98.97	89.81	94.91	97.46	98.97	89.87	94.90	97.43	99.01
100	89.85	94.93	97.51	99.04	89.80	94.91	97.51	99.01	89.75	94.90	97.50	99.02
200	89.85	94.97	97.54	98.97	89.83	95.02	97.53	98.99	89.78	95.02	97.56	99.00
400	89.87	95.03	97.58	99.03	89.92	95.04	97.55	99.02	89.94	95.04	97.50	99.06
Asymptotic CI for ψ_δ based on $\widehat{\Psi}_\delta$												
Value of ψ_δ	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	89.64	94.79	97.34	98.95	89.64	94.83	97.42	98.96	89.69	94.88	97.40	99.00
100	89.75	94.84	97.50	99.03	89.70	94.86	97.44	99.01	89.73	94.91	97.42	99.03
200	89.86	94.92	97.54	98.95	89.83	95.04	97.54	98.98	89.78	95.05	97.57	99.00
400	89.86	95.00	97.59	99.01	89.91	95.01	97.55	99.03	89.93	94.98	97.49	99.05
Asymptotic CI for ψ_Δ based on $\widehat{\Psi}_\Delta$												
Value of ψ_Δ	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	89.99	94.96	97.47	99.00	89.94	95.02	97.54	99.03	89.96	95.01	97.51	99.05
100	89.91	94.96	97.58	99.07	89.90	94.97	97.53	99.03	89.82	94.99	97.54	99.06
200	89.92	95.02	97.56	98.97	89.88	95.06	97.57	98.99	89.84	95.05	97.58	99.00
400	89.88	95.05	97.59	99.03	89.93	95.03	97.55	99.04	89.97	95.01	97.53	99.05

TABLE 3. - *Simulated Average Length (AL) of the exact and asymptotic CIs under the assumption of i.i.d. Normal returns*

Simulated AL of the exact CI for the Sharpe Ratio

Ratio's value	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	0.46800	0.55766	0.63773	0.73289	0.47523	0.56628	0.64759	0.74422	0.49711	0.59234	0.67740	0.77848
100	0.33001	0.39325	0.44971	0.51681	0.33503	0.39920	0.45652	0.52463	0.35018	0.41724	0.47716	0.54835
200	0.23306	0.27770	0.31757	0.36497	0.23658	0.28186	0.32238	0.37047	0.24717	0.29452	0.33685	0.38709
400	0.16469	0.19624	0.22442	0.25790	0.16715	0.19918	0.22778	0.26175	0.17460	0.20809	0.23794	0.27343

Simulated AL of the asymptotic CI for ψ and of the asymptotic CIs for ψ_δ and ψ_Δ based on $\widehat{\Psi}^*_\delta$ and $\widehat{\Psi}^*_\Delta$:
percentage relative variations with respect to the ALs of the exact CIs

Ratio's value	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	0.0088%	0.0088%	0.0089%	0.0090%	0.0309%	0.0310%	0.0312%	0.0316%	0.0920%	0.0921%	0.0925%	0.0929%
100	-0.0001%	0.0027%	0.0026%	0.0025%	0.0176%	0.0134%	0.0133%	0.0134%	0.0482%	0.0440%	0.0437%	0.0437%
200	-0.0001%	-0.0014%	-0.0037%	0.0002%	0.0092%	-0.0032%	0.0107%	0.0069%	0.0163%	0.0144%	0.0260%	0.0227%
400	0.0000%	-0.0002%	-0.0007%	-0.0014%	0.0019%	0.0054%	0.0065%	-0.0016%	-0.0013%	0.0191%	0.0074%	0.0037%

Simulated AL of the asymptotic CI for ψ_δ based on $\widehat{\Psi}_\delta$: percentage relative variations with respect to the AL of the exact CI for ψ_δ

Value of ψ_δ	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	-0.0826%	-0.0826%	-0.0826%	-0.0824%	-0.2903%	-0.2902%	-0.2900%	-0.2895%	-0.8617%	-0.8617%	-0.8613%	-0.8609%
100	-0.0472%	-0.0444%	-0.0445%	-0.0446%	-0.2461%	-0.2503%	-0.2504 %	-0.2503%	-0.8143%	-0.8185%	-0.8188%	-0.8188%
200	-0.0277%	-0.0290%	-0.0312%	-0.0274%	-0.2301%	-0.2426%	-0.2286%	-0.2325%	-0.8087%	-0.8105%	-0.7988%	-0.8022%
400	-0.0179%	-0.0180%	-0.0185%	-0.0193%	-0.2243%	-0.2208%	-0.2197%	-0.2279%	-0.8042%	-0.7837%	-0.7955%	-0.7992%

Simulated AL of the asymptotic CI for ψ_Δ based on $\widehat{\Psi}_\Delta$: percentage relative variation with respect to the AL of the exact CI for ψ_Δ

Value of ψ_Δ	*small*				*medium*				*high*			
100(1−a)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
$n = 50$	0.0011%	0.0011%	0.0011%	0.0012%	0.0039%	0.0040%	0.0043%	0.0047%	0.0125%	0.0125%	0.0129%	0.0133%
100	-0.0057%	-0.0029%	-0.0030%	-0.0031%	-0.0137%	-0.0178%	-0.0179%	-0.0179%	-0.0543%	-0.0584%	-0.0587%	-0.0587%
200	-0.0038%	-0.0052%	-0.0075%	-0.0037%	-0.0239%	-0.0364%	-0.0224%	-0.0263%	-0.0985%	-0.1004%	-0.0887 %	-0.0921%
400	-0.0026%	-0.0028%	-0.0033%	-0.0041%	-0.0317%	-0.0283%	-0.0272%	-0.0353%	-0.1212%	-0.1008%	-0.1125%	-0.1169%

lues of σ are given. In detail, the first part of Table 3 contains only the simulated average lengths of the exact CI for the Sharpe Ratio since the average lengths of the exact CIs for the MAD Ratio and for the MD Ratio can be obtained by multiplying these latter values by $\sqrt{\pi/2}$ and $\sqrt{\pi}/2$, respectively. The length of the asymptotic CIs is evaluated calculating the percentage relative variation with respect to the length of the exact CIs:

$$100\left(\frac{\text{average length of the exact CI} - \text{average length of the asymptotic CI}}{\text{average length of the exact CI}}\right)\%.$$

Note that the percentage relative variation of the CIs (6), (11), and (12) is the same because they differ only in scale.

The simulations show that the asymptotic CIs and the exact ones have similar average lengths. Indeed, all the percentage relative variations are, in absolute value, lower than 1% and, in most cases, they are lower than 0.1%. Concluding, under the assumption of i.i.d. Normal returns, it is possible to assert that a sample size of 50 is sufficient in order to assure that all the asymptotic CIs investigated approximate the exact CIs very well.

7. RESULTS: I.I.D. RETURNS

7.1 *Coverage accuracy of the asymptotic CIs*

First we observe that, as suggested by Remark 5, the features of the asymptotic CIs (14), (16) and (20) are not significantly effected by the value of σ. Then, in the following, we discuss and give the averaged coverages over the different values of σ. The simulated actual coverages of the CI for the Sharpe Ratio, the MAD Ratio, and the MD Ratio are given in Tables 4, 5, and 6, respectively. The results highlight the impact of skewness and fat tails on the actual coverage of the asymptotic CIs. In detail, as suggested by the results obtained sampling from the Student's *t*, the fatter the tails, the worse the coverage accuracy. Regarding the effect of skewness, simulations show that the coverage accuracy improves when sampling from a distribution with positive γ_1. On the contrary, the coverage accuracy worsens when sampling from a distribution with negative γ_1. In the scenarios analyzed, it seems that the presence of fat tails is the element with the greater impact on the coverage accuracy. This fact is suggested mainly by the results obtained sampling from the Skew Normal Distribution. Indeed, in these cases, we observe that the simulated coverage of the asymptotic CIs is quite similar to its nominal value also when $n = 50$ (a similar results is observed when sampling from the Normal distribution).

From Tables 4-6, it turns out that the CIs for the MAD Ratio and the MD Ratio are more accurate than the CI for the Sharpe Ratio. In detail, the coverage accuracy of the CIs (16) and (20) is more robust than the coverage accuracy of the CI (14) with respect to: a) the presence of skewness, b) the presence of fat tails, c) the true value of the performance index.

TABLE 4. - *Averages over the different values of σ of the simulated coverages of the CIs for the Sharpe Ratio*

Value of ψ		*small*				*medium*				*high*			
$100(1-\alpha)\%$	**n**	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
N	*50*	**89.73**	**94.74**	**97.28**	**98.86**	**89.56**	**94.63**	**97.27**	**98.82**	**89.18**	**94.35**	**97.03**	**98.74**
	100	**89.81**	**94.89**	**97.49**	**99.03**	**89.67**	**94.80**	**97.39**	**98.99**	**89.42**	**94.62**	**97.24**	**98.92**
	200	**89.86**	**94.97**	**97.51**	**98.95**	**89.78**	**94.96**	**97.50**	**98.97**	**89.64**	**94.81**	**97.47**	**98.95**
	400	**89.87**	**95.03**	**97.58**	**99.01**	**89.91**	**95.00**	**97.53**	**99.01**	**89.80**	**94.99**	**97.45**	**99.04**
L	*50*	88.02	93.33	96.13	97.97	87.52	92.92	95.89	97.81	86.43	92.04	95.29	97.55
	100	88.83	94.06	**96.77**	98.44	88.70	93.84	96.57	98.39	87.95	93.33	96.24	98.18
	200	**89.50**	**94.49**	**97.15**	**98.81**	**89.30**	**94.37**	**97.05**	**98.69**	**88.96**	**94.11**	**96.79**	**98.51**
	400	**89.69**	**94.78**	**97.39**	**98.86**	**89.45**	**94.73**	**97.35**	**98.88**	**89.25**	**94.48**	**97.16**	**98.79**
$\mathbf{t_5}$	*50*	88.28	93.52	96.40	98.27	88.00	93.32	96.27	98.21	86.75	92.46	95.73	97.85
	100	88.78	**94.08**	**96.81**	**98.51**	88.47	93.85	96.66	**98.48**	87.71	93.21	96.18	98.17
	200	**89.37**	**94.53**	**97.13**	**98.71**	**89.11**	**94.25**	**97.06**	**98.68**	88.52	93.73	96.66	**98.49**
	400	**89.51**	**94.58**	**97.22**	**98.77**	**89.30**	**94.49**	**97.13**	**98.78**	88.83	**94.11**	**96.88**	**98.63**
$\mathbf{t_3}$	*50*	86.08	91.58	94.71	97.11	83.96	90.20	93.92	96.58	79.32	86.30	90.88	94.46
	100	86.79	92.26	95.32	97.62	84.75	90.85	94.47	97.04	79.77	86.72	91.06	94.54
	200	87.99	93.20	96.07	98.07	85.62	91.52	94.83	97.27	80.54	87.35	91.64	95.07
	400	88.31	93.47	96.26	98.18	85.21	91.38	94.73	97.25	80.51	87.28	91.51	95.03
SNL+	*50*	**89.86**	**94.76**	**97.37**	**98.90**	**89.86**	**94.92**	**97.50**	**98.92**	**89.86**	**94.82**	**97.37**	**98.86**
	100	**89.98**	**95.17**	**97.56**	**98.99**	**90.17**	**95.15**	**97.56**	**98.97**	**90.04**	**95.01**	**97.47**	**98.91**
	200	**89.97**	**94.93**	**97.49**	**98.92**	**90.05**	**95.05**	**97.50**	**98.93**	**89.93**	**95.03**	**97.58**	**98.99**
	400	**89.66**	**94.92**	**97.42**	**98.98**	**89.74**	**94.92**	**97.45**	**98.95**	**89.80**	**94.86**	**97.45**	**98.95**
SNH+	*50*	**89.52**	**94.56**	**97.30**	**98.84**	**89.77**	**94.86**	**97.52**	**99.07**	**89.75**	**95.01**	**97.56**	**99.08**
	100	**90.08**	**94.97**	**97.56**	**98.95**	**90.20**	**95.26**	**97.65**	**99.03**	**90.38**	**95.25**	**97.56**	**99.10**
	200	**89.98**	**94.90**	**97.55**	**98.97**	**89.96**	**95.04**	**97.50**	**99.00**	**90.08**	**94.90**	**97.52**	**98.99**
	400	**90.08**	**94.88**	**97.38**	**99.01**	**89.97**	**94.92**	**97.41**	**99.00**	**89.96**	**94.85**	**97.45**	**98.98**
StL+	*50*	88.41	93.64	96.42	98.31	88.65	93.88	96.62	98.42	87.70	93.30	96.19	98.19
	100	**89.19**	**94.21**	**96.84**	**98.61**	**89.45**	**94.43**	**97.01**	**98.68**	88.41	93.84	96.57	98.47
	200	**89.43**	**94.55**	**97.04**	**98.76**	**89.52**	**94.67**	**97.16**	**98.80**	88.76	**94.19**	**96.93**	**98.67**
	400	**89.62**	**94.76**	**97.35**	**98.80**	**89.68**	**94.65**	**97.29**	**98.84**	**89.06**	**94.24**	**96.96**	**98.69**
StH+	*50*	88.17	93.68	96.38	98.28	**89.01**	**94.16**	**96.85**	**98.62**	88.12	93.62	96.56	98.44
	100	**89.34**	**94.26**	**96.92**	**98.54**	**89.75**	**94.67**	**97.18**	**98.77**	88.76	93.87	96.66	98.43
	200	**89.58**	**94.58**	**97.12**	**98.69**	**89.75**	**94.85**	**97.39**	**98.86**	88.85	**94.22**	**96.99**	**98.71**
	400	**89.80**	**94.70**	**97.38**	**98.90**	**89.82**	**94.74**	**97.35**	**98.90**	**89.11**	**94.34**	**96.99**	**98.69**
SNL-	*50*	**89.79**	**94.81**	**97.23**	**98.76**	**89.22**	**94.52**	**96.91**	**98.61**	88.49	**94.07**	96.63	98.39
	100	**89.76**	**94.89**	**97.40**	**98.96**	**89.52**	**94.58**	**97.29**	**98.89**	**89.08**	**94.24**	**97.05**	**98.75**
	200	**89.98**	**95.07**	**97.39**	**98.96**	**89.78**	**94.94**	**97.35**	**98.93**	**89.53**	**94.80**	**97.33**	**98.89**
	400	**90.07**	**95.07**	**97.60**	**99.06**	**89.98**	**95.04**	**97.61**	**99.08**	**89.81**	**94.95**	**97.65**	**99.03**
SNH-	*50*	**89.41**	**94.70**	**97.17**	**98.83**	88.78	**94.19**	**96.93**	**98.70**	88.23	93.52	96.47	98.42
	100	**89.60**	**94.86**	**97.48**	**98.92**	**89.48**	**94.61**	**97.30**	**98.79**	**89.01**	**94.33**	**97.01**	**98.72**
	200	**90.15**	**95.12**	**97.47**	**99.07**	**89.96**	**94.99**	**97.38**	**98.97**	**89.64**	**94.78**	**97.24**	**98.82**
	400	**90.19**	**95.01**	**97.49**	**99.05**	**89.93**	**94.92**	**97.50**	**98.98**	**89.72**	**94.76**	**97.43**	**98.99**
StL-	*50*	87.86	93.06	96.00	98.05	86.80	92.43	95.50	97.78	85.44	91.21	94.86	97.32
	100	**89.04**	94.01	96.73	98.47	88.20	93.57	96.31	98.33	87.06	92.57	95.72	97.98
	200	**89.14**	**94.35**	**96.93**	**98.50**	88.67	93.99	96.62	98.41	87.68	93.26	96.30	98.19
	400	**89.20**	**94.38**	**97.00**	**98.71**	88.72	**94.21**	**96.84**	**98.62**	88.25	93.63	96.57	98.42
StH-	*50*	87.90	93.27	96.28	98.15	86.56	92.35	95.53	97.70	84.87	90.97	94.43	97.07
	100	88.55	93.62	96.58	98.35	87.50	93.08	96.11	98.04	86.18	91.98	95.38	97.61
	200	**89.19**	**94.32**	**96.94**	**98.71**	88.32	93.77	96.56	98.39	87.31	92.92	96.04	98.11
	400	**89.48**	**94.71**	**97.16**	**98.88**	88.82	**94.18**	**97.02**	**98.72**	88.21	93.64	96.58	98.38

Legend: Normal (**N**), Laplace (**L**), Student's t with 5 df ($\mathbf{t}_5$), Student's t with 3 df ($\mathbf{t}_3$), Skew Normal with low positive skewness (**SNL+**), Skew Normal with high positive skewness (**SNH+**), Skew t with low positive skewness (**StL+**), Skew t with high positive skewness (**StH+**), Skew Normal with low negative skewness (**SNL-**), Skew Normal with high negative skewness (**SNH-**), Skew t with low negative skewness (**StL-**), Skew t with high negative skewness (**StH-**).

TABLE 5. - *Averages over the different values of σ of the simulated coverages of the CIs for the MAD Ratio*

Value of ψ_δ		*small*				*medium*				*high*			
100(1-α)%	**n**	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
N	*50*	**89.80**	**94.81**	**97.35**	**98.90**	**89.56**	**94.65**	**97.29**	**98.85**	**89.21**	**94.46**	**97.09**	**98.77**
	100	**89.83**	**94.92**	**97.48**	**99.04**	**89.67**	**94.78**	**97.39**	**98.99**	**89.54**	**94.65**	**97.27**	**98.93**
	200	**89.88**	**94.99**	**97.54**	**98.96**	**89.81**	**95.01**	**97.50**	**98.98**	**89.77**	**94.97**	**97.47**	**98.98**
	400	**89.87**	**95.02**	**97.58**	**99.01**	**89.91**	**95.01**	**97.51**	**99.00**	**89.90**	**94.90**	**97.46**	**99.03**
L	*50*	87.31	92.69	95.59	97.57	86.89	92.35	95.33	97.46	86.76	92.31	95.34	97.53
	100	88.45	93.70	96.46	98.23	88.50	93.60	96.36	98.17	88.42	93.55	96.31	98.17
	200	**89.25**	**94.31**	**96.96**	**98.67**	**89.20**	**94.15**	**96.92**	**98.59**	**89.15**	**94.16**	**96.80**	**98.55**
	400	**89.58**	**94.67**	**97.29**	**98.82**	**89.44**	**94.61**	**97.24**	**98.81**	**89.38**	**94.55**	**97.21**	**98.83**
$\mathbf{t_5}$	*50*	88.36	93.61	96.43	98.32	88.16	93.49	96.34	98.20	87.93	93.36	96.29	98.21
	100	88.95	**94.25**	**96.95**	**98.58**	88.85	94.05	**96.81**	**98.57**	88.68	93.96	96.74	**98.52**
	200	**89.51**	**94.64**	**97.23**	**98.77**	**89.38**	**94.51**	**97.25**	**98.79**	**89.30**	**94.41**	**97.09**	**98.76**
	400	**89.63**	**94.64**	**97.27**	**98.84**	**89.59**	**94.67**	**97.24**	**98.84**	**89.54**	**94.58**	**97.27**	**98.81**
$\mathbf{t_3}$	*50*	86.20	91.79	94.66	96.96	85.81	91.40	94.60	96.88	86.14	91.88	95.16	97.41
	100	87.52	92.86	95.81	97.95	87.08	92.70	95.69	97.75	87.03	92.77	95.92	97.89
	200	88.78	93.94	96.66	**98.50**	88.55	93.69	96.54	98.38	88.50	93.54	96.54	98.38
	400	**89.30**	**94.35**	**96.95**	**98.66**	**89.09**	**94.29**	**96.88**	**98.51**	**89.01**	**94.14**	**96.76**	**98.51**
SNL+	*50*	**89.39**	**94.42**	**97.11**	**98.69**	**89.29**	**94.52**	**97.13**	**98.69**	**89.23**	**94.26**	**96.86**	**98.57**
	100	**89.84**	**94.89**	**97.39**	**98.87**	**89.95**	**94.90**	**97.34**	**98.84**	**89.74**	**94.74**	**97.26**	**98.75**
	200	**89.97**	**94.85**	**97.37**	**98.84**	**89.98**	**94.91**	**97.42**	**98.88**	**89.85**	**94.85**	**97.47**	**98.94**
	400	**89.58**	**94.81**	**97.36**	**98.91**	**89.70**	**94.90**	**97.34**	**98.92**	**89.75**	**94.80**	**97.35**	**98.91**
SNH+	*50*	88.84	94.03	**96.76**	**98.53**	**89.16**	**94.28**	**96.98**	**98.70**	**89.08**	**94.35**	**96.95**	**98.65**
	100	**89.85**	**94.71**	**97.28**	**98.80**	**89.92**	**94.80**	**97.43**	**98.86**	**89.97**	**94.84**	**97.28**	**98.92**
	200	**89.69**	**94.73**	**97.36**	**98.88**	**89.72**	**94.84**	**97.37**	**98.90**	**89.89**	**94.72**	**97.36**	**98.87**
	400	**89.97**	**94.79**	**97.25**	**98.93**	**89.86**	**94.88**	**97.31**	**98.94**	**89.88**	**94.74**	**97.37**	**98.93**
StL+	*50*	88.33	93.61	96.43	98.24	88.26	93.64	96.41	98.28	88.42	93.68	96.49	98.40
	100	**89.27**	**94.27**	**96.90**	**98.66**	**89.25**	**94.26**	**96.94**	**98.63**	**89.05**	**94.23**	**96.95**	**98.68**
	200	**89.50**	**94.60**	**97.07**	**98.77**	**89.50**	**94.63**	**97.11**	**98.77**	**89.62**	**94.63**	**97.10**	**98.78**
	400	**89.75**	**94.83**	**97.33**	**98.86**	**89.74**	**94.86**	**97.33**	**98.87**	**89.67**	**94.75**	**97.41**	**98.84**
StH+	*50*	87.92	93.56	96.29	98.23	88.23	93.70	96.40	98.29	88.62	93.96	**96.76**	**98.55**
	100	**89.30**	**94.27**	**96.95**	**98.59**	**89.30**	**94.42**	**96.98**	**98.61**	**89.35**	**94.48**	**97.09**	**98.70**
	200	**89.67**	**94.59**	**97.14**	**98.72**	**89.67**	**94.60**	**97.17**	**98.76**	**89.74**	**94.73**	**97.29**	**98.85**
	400	**89.78**	**94.71**	**97.35**	**98.89**	**89.71**	**94.75**	**97.41**	**98.93**	**89.77**	**94.89**	**97.31**	**98.93**
SNL-	*50*	**89.89**	**94.93**	**97.33**	**98.83**	**89.40**	**94.68**	**97.11**	**98.74**	**89.05**	**94.30**	**96.86**	**98.58**
	100	**89.79**	**94.95**	**97.39**	**98.95**	**89.59**	**94.69**	**97.31**	**98.83**	**89.45**	**94.43**	**97.11**	**98.75**
	200	**89.95**	**95.07**	**97.39**	**98.98**	**89.90**	**95.00**	**97.44**	**98.93**	**89.74**	**94.85**	**97.47**	**98.91**
	400	**90.07**	**95.06**	**97.59**	**99.07**	**90.02**	**95.09**	**97.56**	**99.05**	**89.98**	**95.02**	**97.57**	**99.03**
SNH-	*50*	**89.56**	**94.81**	**97.31**	**98.92**	**89.13**	**94.34**	**97.10**	**98.74**	88.79	94.00	**96.80**	**98.56**
	100	**89.69**	**94.90**	**97.47**	**98.92**	**89.62**	**94.74**	**97.38**	**98.79**	**89.42**	**94.51**	**97.23**	**98.78**
	200	**90.16**	**95.14**	**97.48**	**99.07**	**90.00**	**95.00**	**97.44**	**98.96**	**89.75**	**94.86**	**97.30**	**98.86**
	400	**90.18**	**95.04**	**97.50**	**99.05**	**89.95**	**94.95**	**97.56**	**99.02**	**89.75**	**94.88**	**97.47**	**98.94**
StL-	*50*	87.95	93.18	96.11	98.09	87.47	92.96	95.92	97.92	87.20	92.77	95.76	97.90
	100	**89.28**	**94.23**	**96.91**	**98.53**	88.90	**94.07**	**96.74**	**98.49**	88.54	93.78	96.55	98.43
	200	**89.37**	**94.57**	**97.11**	**98.68**	**89.28**	**94.31**	**96.96**	**98.62**	**89.07**	**94.21**	**96.93**	**98.54**
	400	**89.34**	**94.60**	**97.14**	**98.78**	**89.24**	**94.48**	**97.11**	**98.82**	**89.24**	**94.33**	**97.11**	**98.75**
StH-	*50*	88.15	93.52	96.39	98.29	87.61	93.04	96.05	98.03	87.30	92.72	95.75	97.86
	100	88.82	93.89	**96.79**	**98.49**	88.64	93.78	96.62	98.39	88.24	93.61	96.42	98.34
	200	**89.38**	**94.53**	**97.16**	**98.81**	**89.19**	**94.34**	**97.12**	**98.74**	88.90	**94.26**	**96.91**	**98.64**
	400	**89.74**	**94.85**	**97.29**	**98.97**	**89.43**	**94.71**	**97.26**	**98.92**	**89.34**	**94.67**	**97.25**	**98.85**

Legend: Normal (**N**), Laplace (**L**), Student's t with 5 df ($\mathbf{t_5}$), Student's t with 3 df ($\mathbf{t_3}$), Skew Normal with low positive skewness (**SNL+**), Skew Normal with high positive skewness (**SNH+**), Skew t with low positive skewness (**StL+**), Skew t with high positive skewness (**StH+**), Skew Normal with low negative skewness (**SNL-**), Skew Normal with high negative skewness (**SNH-**), Skew t with low negative skewness (**StL-**), Skew t with high negative skewness (**StH-**).

TABLE 6. - *Averages over the different values of σ of the simulated coverages of the CIs for the MD Ratio*

Value of ψ_δ		*small*				*medium*				*high*			
100(1-α)%	**n**	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
N	*50*	**89.77**	**94.79**	**97.33**	**98.91**	**89.67**	**94.76**	**97.34**	**98.86**	**89.40**	**94.57**	**97.18**	**98.81**
	100	**89.84**	**94.90**	**97.50**	**99.05**	**89.71**	**94.86**	**97.40**	**99.00**	**89.58**	**94.71**	**97.28**	**98.94**
	200	**89.87**	**94.99**	**97.53**	**98.96**	**89.85**	**94.98**	**97.54**	**98.98**	**89.72**	**94.93**	**97.50**	**98.98**
	400	**89.86**	**95.02**	**97.58**	**99.01**	**89.91**	**95.03**	**97.53**	**99.02**	**89.95**	**94.97**	**97.48**	**99.05**
L	*50*	88.06	93.39	96.15	97.96	87.85	93.15	96.02	97.92	87.58	92.93	95.88	97.90
	100	88.88	**94.10**	**96.79**	98.46	**88.96**	94.02	96.69	98.43	88.71	93.88	96.62	98.40
	200	**89.50**	**94.52**	**97.16**	**98.82**	**89.45**	**94.40**	**97.11**	**98.73**	**89.41**	**94.37**	**96.96**	**98.66**
	400	**89.73**	**94.80**	**97.39**	**98.86**	**89.55**	**94.76**	**97.38**	**98.89**	**89.41**	**94.66**	**97.29**	**98.90**
$\mathbf{t_5}$	*50*	88.46	93.71	96.54	98.35	88.43	93.66	96.51	98.34	88.15	93.57	96.44	98.31
	100	**88.98**	**94.26**	**96.97**	**98.62**	88.88	**94.17**	**96.87**	**98.64**	88.77	94.03	**96.78**	**98.57**
	200	**89.50**	**94.66**	**97.24**	**98.78**	**89.41**	**94.48**	**97.26**	**98.75**	**89.36**	**94.45**	**97.08**	**98.75**
	400	**89.64**	**94.66**	**97.27**	**98.83**	**89.60**	**94.66**	**97.23**	**98.84**	**89.49**	**94.58**	**97.21**	**98.82**
$\mathbf{t_3}$	*50*	86.61	92.13	95.12	97.37	86.33	92.04	95.17	97.31	86.15	92.02	95.22	97.39
	100	87.67	93.07	96.02	98.08	87.21	92.99	95.82	97.92	87.07	92.69	95.83	97.89
	200	88.83	93.98	96.68	**98.53**	88.62	93.74	96.58	98.38	88.28	93.42	96.48	98.31
	400	**89.29**	**94.32**	**96.90**	**98.65**	**88.99**	**94.14**	**96.85**	**98.51**	88.66	94.02	96.66	**98.49**
SNL+	*50*	**89.88**	**94.79**	**97.44**	**98.88**	**89.67**	**94.68**	**97.44**	**98.83**	**89.57**	**94.66**	**97.13**	**98.83**
	100	**89.99**	**95.13**	**97.60**	**98.98**	**90.10**	**95.11**	**97.47**	**98.97**	**89.99**	**94.88**	**97.34**	**98.95**
	200	**90.01**	**94.94**	**97.46**	**98.91**	**90.03**	**95.06**	**97.39**	**98.93**	**89.98**	**94.96**	**97.47**	**98.94**
	400	**89.68**	**94.90**	**97.42**	**98.97**	**89.79**	**94.86**	**97.39**	**98.93**	**89.88**	**94.80**	**97.38**	**98.90**
SNH+	*50*	**89.56**	**94.59**	**97.31**	**98.89**	**89.57**	**94.69**	**97.39**	**98.98**	**89.46**	**94.66**	**97.43**	**98.92**
	100	**90.04**	**94.99**	**97.58**	**98.96**	**90.16**	**95.12**	**97.60**	**99.01**	**90.17**	**95.03**	**97.55**	**99.06**
	200	**89.94**	**94.91**	**97.56**	**98.97**	**89.92**	**94.95**	**97.50**	**98.98**	**89.89**	**94.94**	**97.50**	**98.98**
	400	**90.10**	**94.88**	**97.38**	**98.98**	**90.07**	**94.95**	**97.41**	**98.96**	**89.89**	**94.89**	**97.49**	**99.00**
StL+	*50*	88.45	93.76	96.49	98.34	88.64	93.90	96.68	98.43	88.79	**94.10**	96.73	**98.53**
	100	**89.30**	**94.29**	**96.96**	**98.67**	**89.48**	**94.30**	**96.99**	**98.68**	**89.37**	**94.29**	**97.07**	**98.75**
	200	**89.53**	**94.63**	**97.10**	**98.80**	**89.62**	**94.65**	**97.14**	**98.81**	**89.66**	**94.73**	**97.20**	**98.85**
	400	**89.74**	**94.84**	**97.37**	**98.85**	**89.77**	**94.78**	**97.36**	**98.86**	**89.71**	**94.77**	**97.34**	**98.89**
StH+	*50*	88.13	93.67	96.45	98.31	88.70	94.04	96.68	98.46	**89.02**	**94.39**	**97.09**	**98.66**
	100	**89.37**	**94.35**	**96.97**	**98.58**	**89.53**	**94.53**	**97.07**	**98.66**	**89.50**	**94.73**	**97.17**	**98.69**
	200	**89.64**	**94.61**	**97.14**	**98.72**	**89.74**	**94.68**	**97.22**	**98.84**	**89.70**	**94.78**	**97.31**	**98.89**
	400	**89.82**	**94.71**	**97.33**	**98.89**	**89.81**	**94.80**	**97.40**	**98.98**	**90.00**	**94.95**	**97.33**	**98.98**
SNL-	*50*	**89.84**	**94.82**	**97.31**	**98.83**	**89.37**	**94.70**	**97.10**	**98.73**	**89.16**	**94.39**	**96.86**	**98.55**
	100	**89.79**	**94.91**	**97.41**	**98.95**	**89.67**	**94.71**	**97.35**	**98.88**	**89.48**	**94.44**	**97.17**	**98.84**
	200	**89.98**	**95.08**	**97.40**	**98.98**	**89.81**	**95.01**	**97.41**	**98.95**	**89.77**	**94.83**	**97.42**	**98.96**
	400	**90.06**	**95.09**	**97.59**	**99.05**	**90.03**	**95.07**	**97.57**	**99.08**	**89.96**	**94.97**	**97.63**	**99.06**
SNH-	*50*	**89.52**	**94.77**	**97.21**	**98.87**	**89.18**	**94.34**	**97.08**	**98.77**	88.81	93.99	**96.75**	**98.62**
	100	**89.62**	**94.89**	**97.47**	**98.92**	**89.56**	**94.71**	**97.36**	**98.82**	**89.37**	**94.60**	**97.25**	**98.78**
	200	**90.13**	**95.10**	**97.45**	**99.06**	**89.88**	**95.05**	**97.43**	**98.98**	**89.80**	**94.91**	**97.35**	**98.88**
	400	**90.19**	**95.04**	**97.48**	**99.04**	**90.04**	**95.00**	**97.51**	**98.98**	**89.78**	**94.93**	**97.48**	**98.95**
StL-	*50*	88.09	93.22	96.19	98.15	87.53	93.09	96.05	98.11	87.24	92.89	95.90	97.94
	100	**89.29**	**94.26**	**96.93**	**98.56**	88.94	**94.12**	96.73	**98.53**	88.51	93.77	96.53	98.44
	200	**89.41**	**94.56**	**97.10**	**98.65**	**89.38**	**94.32**	**96.96**	**98.63**	**88.99**	**94.20**	**96.93**	**98.56**
	400	**89.32**	**94.56**	**97.14**	**98.77**	**89.27**	**94.53**	**97.08**	**98.80**	**89.09**	**94.32**	**97.09**	**98.74**
StH-	*50*	88.19	93.59	96.49	98.31	87.66	93.23	96.14	98.13	87.37	92.85	95.82	97.91
	100	88.82	93.86	**96.81**	98.48	88.60	93.71	96.64	98.44	88.19	93.61	96.47	98.36
	200	**89.40**	**94.51**	**97.13**	**98.80**	**89.14**	**94.35**	**97.08**	**98.73**	88.93	**94.17**	**96.94**	**98.64**
	400	**89.73**	**94.86**	**97.25**	**98.98**	**89.49**	**94.71**	**97.30**	**98.91**	**89.30**	**94.62**	**97.20**	**98.82**

Legend: Normal (**N**), *Laplace* (**L**), Student's t with 5 df ($\mathbf{t_5}$), Student's t with 3 df ($\mathbf{t_3}$), Skew Normal with low positive skewness (**SNL+**), Skew Normal with high positive skewness (**SNH+**), Skew t with low positive skewness (**StL+**), Skew t with high positive skewness (**StH+**), Skew Normal with low negative skewness (**SNL-**), Skew Normal with high negative skewness (**SNH-**), Skew t with low negative skewness (**StL-**), Skew t with high negative skewness (**StH-**).

As regards the impact of the true value of the index, we observe that, generally, the greater the true value of the index, the worse the coverage accuracy. However, when ψ, ψ_δ, and ψ_Δ increase, the variations in the simulated coverage of the CI (16), and (20) are relatively small, while the changes in the simulated coverages of CI (14) are greater. This effect is more evident when sampling from the Laplace and Student's t distributions suggesting that the changes are greater when the tails are fatter. In general, the CI for the MAD Ratio seems to have better coverage accuracy (even if the differences with the CI for the MD Ratio are quite small).

In order to determine the minimum sample size required for the sufficient precision of the asymptotic CIs, we introduce the following criterion. It is well known that the t distribution approaches the Normal distribution when the df increase. Further, it is common to retain that the t distribution with 30 df is approximated by the Normal distribution very well. As a consequence, when sampling from the Normal distribution, the asymptotic CI

$$(\bar{X} - z_{1-\alpha/2}\sqrt{S^2/n}; \bar{X} + z_{1-\alpha/2}\sqrt{S^2/n}) \tag{32}$$

is considered accurate if $n \geq 30$. The actual coverages of the above CI when $n = 30$ are given in Table 7 and, effectively, they are quite similar to their nominal values.

Then, we think it is reasonable to take the values given in Table 7 as a benchmark and, in the following, we assert that the simulated coverage of a asymptotic CI is sufficiently close to its nominal value $(1 - \alpha)$ if it belongs to the interval $(1 - \alpha - \epsilon_\alpha; 1 - \alpha + \epsilon_\alpha)$, where ϵ_α is defined in Table 7.

The simulated coverages belonging to this interval are written in bold in Tables 4-6. The application of the above criterion leads to the minimum sample sizes given in Table 8.

It can be noted that, with the exception of the Student's t with 3 df, a sample size of 200 is always sufficient in order to assure a good coverage accuracy of the CIs for the MAD Ratio and for the MD Ratio. Differently, a sample size of 400 is not always sufficient for the CIs on the Sharpe Ratio as it can be observed in the case of the Skew t distribution with negative γ_1. Regarding the Student's t with 3 df, as suggested by Remark 4, the coverage accuracy of the asymptotic CI (14) is quite bad, while $n = 400$ turns out to be sufficient for the CI (20) and for the CI (16) with low and intermediate value of ψ_Δ. Finally, it is worthwhile to note that the scenarios concerning the Skew t distribution are the more realistic because the empirical distribu-

TABLE 7. - *Comparison between the nominal and actual coverage probabilities of the asymptotic CI (32)*

Nominal Coverage: $(1 - \alpha)$	0.9	0.95	0.975	0.99
Actual Coverage: a_α	0.8896	0.9407	0.9674	0.9848
$\epsilon_\alpha = \lvert(1 - \alpha) - a_\alpha\rvert$	0.0104	0.0093	0.0076	0.0052
$(1 - \alpha) + \epsilon_\alpha$	0.9104	0.9593	0.9826	0.9952
$(1 - \alpha) - \epsilon_\alpha$	0.8896	0.9407	0.9674	0.9848

TABLE 8. - *Minimum sample sizes assuring a sufficient precision of the asymptotic CIs for the three ratios under the assumption of i.i.d. returns*

	Sharpe Ratio			**MAD Ratio**			**MD Ratio**		
	small	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
N	50	50	50	50	50	50	50	50	50
L	200	200	200	200	200	200	100	200	200
$\mathbf{t_5}$	100	200	400	100	100	200	100	100	200
$\mathbf{t_3}$	—	—	—	400	400	400	400	400	>400
SNL+	50	50	50	50	50	50	50	50	50
SNH+	50	50	50	50	50	50	50	50	50
StL+	100	100	200	100	100	100	100	100	100
StH+	100	50	200	100	100	100	100	100	50
SNL-	50	50	100	50	50	50	50	50	50
SNH-	50	50	100	50	50	100	50	50	100
StL-	200	400	>400	100	100	200	100	100	200
StH-	200	400	>400	200	200	200	200	200	200

Legend: Normal (**N**), Laplace (**L**), Student's t with 5 df ($\mathbf{t_5}$), Student's t with 3 df ($\mathbf{t_3}$), Skew Normal with low positive skewness (**SNL+**), Skew Normal with high positive skewness (**SNH+**), Skew t with low positive skewness (**StL+**), Skew t with high positive skewness (**StH+**), Skew Normal with low negative skewness (**SNL-**), Skew Normal with high negative skewness (**SNH-**), Skew t with low negative skewness (**StL-**), Skew t with high negative skewness (**StH-**).

tions of the financial assets are generally characterized by the presence of fat tails and skewness. Furthermore, these are the cases in which the performances of the asymptotic CIs (16) and (20) are quite better than the performance of the CI (14).

7.2 *Length of the asymptotic CIs*

Once again, we observe that the value of σ does not impact significantly on the Average Length of the CIs. Consequently, the values given in Table 9 and Table 10 are averages calculated over the different values of σ. In detail, in Table 9, the Average Lengths (ALs) obtained sampling from the Normal distribution are given. The ALs obtained under the other scenarios are evaluated by the percentage relative variation with respect to the lengths obtained sampling from the Normal:

$$100\left(\frac{\text{AL sampling from a given distribution} - \text{AL sampling from the Normal}}{\text{AL sampling from the Normal}}\right)\%.$$

The values of the above index are given in Table 10. Observe that, in Table 10, the values of $(1-\alpha)$ are not specified since the length of the CIs is proportional to the quantiles of the standard Normal distribution. As a consequence, the percentage relative variations are the same for all the values of $(1-\alpha)$.

Simulations show that the features of the ALs of the CIs (16) and (20) are quite different from those of the CI (14). In detail, as expected, the CIs (16) and (20) tend to be larger if the parent distribution has tails heavier than the Normal. On the contrary, the CI (14) tends to be larger if the parent distributions have tails heavier than the Normal only for intermediate and large values of ψ. Regarding the effect of skewness, evidence from the Skew Normal distribution suggests that the AL of CIs (14), (16) and (20) increases (decreases) when the parent distribution becomes more negatively (positively) skewed. When sampling from the Skew t distribution (which exhibits both skewness and fat tails) we observe that, in a sense, the effect of skewness prevails on the effect of fat tails in the case of the CI for the Sharpe Ratio. For the other two indices, the effect of fat tails prevails. In detail, when sampling from StL+ and StH+, the CI (14) tends to be shorter than in the Normal case, also for intermediate and large values of ψ, while the CIs (16) and (20) are larger than in the Normal case.

7.3 *Evaluation of the performances of the bias corrected estimators*

In order to compare the bias and the efficiency of the natural estimators with those of the approximately unbiased ones, we compute, for each scenario, the simulated bias and MSE of $\widehat{\Psi}$, $\widehat{\Psi}_\delta$, $\widehat{\Psi}_\Delta$, $\widehat{\Psi}'_u$, $\widehat{\Psi}_{\delta u}$, and $\widehat{\Psi}_{\Delta u}$. In Tables 11 and 12 we give the average, over the values of σ and over the 12 distribution considered, of the simulated bias and MSE. In order to ease comparisons, in these tables, the following gain/loss indices are given:

TABLE 9. - *Averages over the different values of σ of the simulated AL of the asymptotic CIs for ψ, ψ_δ, and ψ_Δ when sampling from the Normal distribution*

Sharpe Ratio Value of ψ	*small*				*medium*				*high*			
100(1-α)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
n = 50	0.4675	0.5571	0.6371	0.7321	0.4734	0.5641	0.6451	0.7414	0.4917	0.5858	0.6700	0.7699
100	0.3299	0.3931	0.4496	0.5166	0.3345	0.3986	0.4558	0.5238	0.3483	0.4151	0.4747	0.5455
200	0.2330	0.2777	0.3176	0.3649	0.2364	0.2817	0.3221	0.3702	0.2465	0.2938	0.3359	0.3861
400	0.1647	0.1962	0.2244	0.2579	0.1671	0.1991	0.2277	0.2616	0.1743	0.2077	0.2376	0.2730
MAD Ratio Value of ψ_δ	*small*				*medium*				*high*			
100(1-α)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
n = 50	0.5902	0.7032	0.8042	0.9242	0.5977	0.7122	0.8144	0.9360	0.6262	0.7462	0.8533	0.9806
100	0.4148	0.4943	0.5653	0.6496	0.4211	0.5017	0.5738	0.6594	0.4418	0.5264	0.6020	0.6918
200	0.2926	0.3486	0.3987	0.4582	0.2973	0.3542	0.4051	0.4655	0.3121	0.3719	0.4253	0.4887
400	0.2066	0.2462	0.2815	0.3235	0.2099	0.2502	0.2861	0.3288	0.2204	0.2627	0.3004	0.3452
MD Ratio Value of ψ_Δ	*small*				*medium*				*high*			
100(1-α)%	90%	95%	97.5%	99%	90%	95%	97.5%	99%	90%	95%	97.5%	99%
n = 50	0.4127	0.4918	0.5624	0.6464	0.4189	0.4992	0.5709	0.6560	0.4379	0.5218	0.5967	0.6857
100	0.2918	0.3476	0.3976	0.4569	0.2962	0.3529	0.4036	0.4638	0.3096	0.3690	0.4219	0.4849
200	0.2063	0.2458	0.2811	0.3231	0.2094	0.2496	0.2854	0.3280	0.2190	0.2609	0.2984	0.3429
400	0.1459	0.1738	0.1988	0.2284	0.1481	0.1764	0.2018	0.2319	0.1548	0.1844	0.2109	0.2424

TABLE 10. - *Percentage relative variation of the AL of the asymptotic CIs with respect to the ALs obtained sampling from the Normal distribution*

		Sharpe Ratio			**MAD Ratio**			**MD Ratio**		
		small	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
L	$n = 50$	-1.79	-0.45	3.29	7.90	9.36	13.54	3.54	5.11	9.41
	100	-0.92	0.67	5.07	10.27	11.70	15.63	4.97	6.53	10.79
	200	-0.44	1.38	6.33	11.49	12.88	16.68	5.68	7.26	11.55
	400	-0.18	1.80	7.11	12.18	13.61	17.41	6.06	7.68	12.02
$\mathbf{t_5}$	$n = 50$	-1.62	-0.63	2.25	4.92	5.69	7.97	2.32	3.30	6.06
	100	-0.99	0.44	4.44	6.59	7.38	9.63	3.62	4.68	7.65
	200	-0.54	1.36	6.50	7.49	8.32	10.61	4.37	5.53	8.70
	400	-0.27	2.02	8.16	8.00	8.84	11.15	4.80	5.99	9.26
$\mathbf{t_3}$	$n = 50$	-4.01	-1.22	6.56	9.97	12.50	19.83	5.45	8.39	16.37
	100	-3.04	1.05	12.42	14.96	17.34	24.44	9.20	12.23	20.79
	200	-2.07	4.09	20.05	18.41	21.02	28.28	11.98	15.39	24.71
	400	-1.48	7.36	29.13	20.83	23.49	30.88	14.03	17.65	27.58
SNL+	$n = 50$	-0.90	-4.58	-8.41	-0.57	-4.97	-9.47	-0.40	-4.52	-8.79
	100	-0.98	-5.02	-9.14	-0.51	-5.10	-9.77	-0.36	-4.74	-9.29
	200	-1.06	-5.22	-9.43	-0.55	-5.16	-9.84	-0.39	-4.84	-9.46
	400	-1.09	-5.29	-9.52	-0.56	-5.17	-9.84	-0.41	-4.87	-9.50
SNH+	$n = 50$	-1.14	-6.76	-12.70	-0.84	-7.68	-14.78	-0.12	-6.57	-13.42
	100	-1.37	-7.36	-13.57	-1.04	-8.02	-15.24	-0.25	-6.95	-14.08
	200	-1.50	-7.69	-14.03	-1.11	-8.17	-15.46	-0.30	-7.13	-14.41
	400	-1.57	-7.83	-14.20	-1.16	-8.21	-15.49	-0.34	-7.20	-14.52
StL+	$n = 50$	-2.24	-3.99	-3.83	4.24	1.71	0.77	1.80	-0.17	-0.32
	100	-1.82	-3.77	-2.90	5.79	2.97	1.69	3.02	0.75	0.42
	200	-1.55	-3.56	-1.88	6.64	3.66	2.22	3.71	1.27	0.87
	400	-1.39	-3.24	-0.56	7.21	4.21	2.77	4.16	1.70	1.37
StH+	$n = 50$	-2.74	-7.35	-10.01	3.83	-2.18	-6.54	1.79	-3.35	-6.68
	100	-2.57	-7.81	-9.95	5.24	-1.23	-6.09	2.90	-2.80	-6.56
	200	-2.47	-8.09	-9.35	6.17	-0.59	-5.74	3.64	-2.40	-6.38
	400	-2.42	-8.10	-8.35	6.62	-0.24	-5.50	4.02	-2.16	-6.20
SNL-	$n = 50$	1.03	4.82	8.95	1.72	6.14	10.73	1.77	5.94	10.32
	100	1.13	5.23	9.67	1.84	6.39	11.10	1.89	6.23	10.78
	200	1.13	5.40	10.03	1.86	6.46	11.21	1.91	6.33	10.98
	400	1.13	5.48	10.20	1.83	6.45	11.23	1.90	6.36	11.05
SNH-	$n = 50$	1.74	7.31	13.31	2.50	9.15	15.96	3.11	9.35	15.80
	100	1.74	7.75	14.20	2.55	9.35	16.33	3.18	9.65	16.36
	200	1.69	7.92	14.62	2.45	9.28	16.27	3.13	9.70	16.50
	400	1.69	8.06	14.90	2.45	9.35	16.42	3.15	9.80	16.68
StL-	$n = 50$	-0.78	3.02	8.70	5.97	10.04	15.56	3.40	7.37	13.08
	100	-0.01	4.80	12.05	7.72	12.07	17.81	4.78	9.13	15.32
	200	0.46	6.17	14.93	8.62	13.12	19.01	5.55	10.13	16.64
	400	0.89	7.44	17.52	9.19	13.84	19.88	6.05	10.84	17.61
StH-	$n = 50$	0.26	6.82	15.26	7.35	14.74	23.54	5.02	11.97	20.58
	100	0.95	9.03	19.50	9.02	16.87	26.06	6.39	13.97	23.30
	200	1.59	11.00	23.35	10.00	18.12	27.56	7.28	15.25	25.05
	400	2.00	12.64	26.77	10.64	18.98	28.65	7.84	16.09	26.22

Legend: Laplace (**L**), Student's t with 5 df ($\mathbf{t_5}$), Student's t with 3 df ($\mathbf{t_3}$), Skew Normal with low positive skewness (**SNL+**), Skew Normal with high positive skewness (**SNH+**), Skew t with low positive skewness (**StL+**), Skew t with high positive skewness (**StH**), Skew Normal with low negative skewness (**SNL-**), Skew Normal with high negative skewness (**SNH-**), Skew t with low negative skewness (**StL-**), Skew t with high negative skewness (**StH-)**.

$$100\left(\frac{\text{MSE of the approx. unbiased est.} - \text{MSE of the natural est.}}{\text{MSE of the natural est.}}\right)\%, \qquad (33)$$

$$100\left(\frac{|\text{bias of the approx. unbiased est.}| - |\text{bias of the natural est.}|}{\text{true value of the ratio}}\right)\%. \qquad (34)$$

In computing the value of the gain/loss index (34), the denominator is set equal to the average, over the 12 distributions, of the true (small, medium or high) value of the ratios. In particular, we use the following values: averaged Sharpe Ratio ($small = 0.05$, $medium = 0.25$, $high = 0.5$); averaged MAD Ratio ($small = 0.0671$, $medium = 0.3355$, $high = 0.6711$); averaged MD Ratio ($small = 0.0465$, $medium = 0.2323$, $high = 0.4646$).

Simulations show that the approximately unbiased estimator generally has a lower bias than natural estimators. However, the gain/loss indices in Table 11 show that the bias reduction is negligible. Furthermore, Table 12 shows that the increase in the variability of the approximatively unbiased estimators (due to the estimation of the bias factors) compensates the bias reduction so that the MSE of the natural estimators turns out to be (a little) lower than MSE of the approximatively unbiased ones. Con-

TABLE 11. - *Comparison between the bias of the natural estimators and that of the approximately unbiased ones*

Bias of the Sharpe Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (34)**		
Value of ψ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
$n = 50$	0.00182	0.00990	0.01999	0.00143	0.00796	0.01612	-0.79%	-0.78%	-0.78%
100	0.00141	0.00603	0.01181	0.00116	0.00470	0.00912	-0.50%	-0.53%	-0.54%
200	0.00049	0.00325	0.00669	0.00033	0.00244	0.00507	-0.33%	-0.33%	-0.33%
400	0.00040	0.00203	0.00406	0.00032	0.00159	0.00319	-0.17%	-0.17%	-0.17%
Bias of the MAD Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (34)**		
Value of ψ_δ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
$n = 50$	0.00135	0.00799	0.01629	0.00086	0.00288	0.00540	-0.72%	-1.52%	-1.62%
100	0.00115	0.00438	0.00842	0.00075	0.00168	0.00284	-0.60%	-0.80%	-0.83%
200	0.00020	0.00187	0.00397	-0.00004	0.00051	0.00119	-0.23%	-0.41%	-0.41%
400	0.00028	0.00112	0.00218	0.00014	0.00040	0.00071	-0.21%	-0.22%	-0.22%
Bias of the MD Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (34)**		
Value of ψ_Δ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
$n = 50$	0.00065	0.00392	0.00800	-0.00008	0.00028	0.00072	-1.24%	-1.57%	-1.57%
100	0.00064	0.00224	0.00424	0.00027	0.00036	0.00047	-0.81%	-0.81%	-0.81%
200	0.00002	0.00087	0.00192	-0.00017	-0.00012	-0.00006	0.33%	-0.32%	-0.40%
400	0.00011	0.00054	0.00108	0.00001	0.00004	0.00008	-0.20%	-0.21%	-0.22%

TABLE 12. - *Comparison between the MSE of the natural estimators and that of the approximately unbiased ones*

MSE of the Sharpe Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (33)**		
Value of ψ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
n = 50	0.0217	0.0234	0.0286	0.0219	0.0236	0.0288	1.10%	0.94%	0.56%
100	0.0111	0.0120	0.0149	0.0112	0.0121	0.0150	0.98%	0.92%	0.78%
200	0.0059	0.0064	0.0080	0.0059	0.0065	0.0081	0.89%	0.90%	0.92%
400	0.0031	0.0034	0.0044	0.0031	0.0035	0.0044	0.88%	0.94%	1.06%
MSE of the MAD Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (33)**		
Value of ψ_δ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
n = 50	0.0382	0.0404	0.0473	0.0381	0.0402	0.0468	-0.22%	-0.41%	-0.91%
100	0.0198	0.0209	0.0243	0.0198	0.0209	0.0243	0.36%	0.23%	-0.06%
200	0.0106	0.0112	0.0130	0.0107	0.0112	0.0131	0.52%	0.43%	0.19%
400	0.0057	0.0060	0.0070	0.0057	0.0060	0.0070	0.60%	0.48%	0.20%
MSE of the MD Ratio estimators									
	Natual Est.			**Appr. Unbiased Est.**			**Index (33)**		
Value of ψ_Δ	*small*	*medium*	*high*	*small*	*medium*	*high*	*small*	*medium*	*high*
n = 50	0.0180	0.0191	0.0223	0.0180	0.0190	0.0222	-0.32%	-0.37%	-0.51%
100	0.0094	0.0099	0.0115	0.0094	0.0099	0.0116	0.17%	0.13%	0.06%
200	0.0050	0.0053	0.0062	0.0050	0.0053	0.0062	0.28%	0.26%	0.19%
400	0.0027	0.0028	0.0033	0.0027	0.0028	0.0033	0.31%	0.26%	0.14%

cluding, in the scenarios considered, the natural estimators can be considered more efficient than the approximately unbiased ones.

8. CONCLUSIONS

In this paper we study the inferential aspects of the Sharpe Ratio, MD Ratio, and MAD Ratio.

Under the assumption of i.i.d. Normal returns, we introduce an exact CI for each ratio and we study the coverage accuracy of the asymptotic confidence intervals defined on the basis of the results in Jobson and Korkie (1981), Gastwirth (1974), and Hoeffding (1948). In a brief simulation study, we observe that all these asymptotic CIs approximate the exact ones very well starting from the relatively small sample size of 50. Moreover, we recall that the natural estimators of the Sharpe Ratio is biased and we improve the approximation of the bias factor given in Jobson and Korkie (1981).

Under the assumption if i.i.d. returns, we recall the asymptotic confidence interval for the Sharpe Ratio stemming from the result of Lo (2002) and we introduce asymptotic CIs for the MAD Ratio and for the MD Ratio. We investigate the coverage accuracy of these asymptotic CIs in a wide simulation study. We deduce that the asymptotic CIs for the MD Ratio and for the MAD Ratio have quite better features than the CI for the Sharpe Ratio. In detail, we highlight that the CIs for the MD Ratio and for the MAD Ratio are more robust with respect to the presence of skewness and fat tails in the parent distribution, especially when the latter does not possess all the moments (such as the Student's t or the Skew t). Since the only difference between the three indicators is the variability measure used to quantify the risk, the difference in the performances of the asymptotic CIs can be attributed to the different features of the estimators S^2, $\hat{\delta}$ and $\hat{\Delta}$ and their different covariance with the sample mean $\bar{X}$. In detail, the evidence we obtain can be partially motivated by the fact that some desirable properties of S^2 are based on stronger assumptions than those required by $\hat{\Delta}$ and $\hat{\delta}$. For example, $\hat{\Delta}$ and $\hat{\delta}$ are weakly consistent (i.e. they converge in probability to the true value of Δ and δ, respectively) if the parent distribution has the first moment; while the existence of the second moment is required for the weak consistency of S^2. Moreover, $\hat{\Delta}$ and $\hat{\delta}$ are asymptotically normally distributed if the parent distribution has second moment (as shown in Theorems 2 and 3). On the contrary, the existence of the fourth moment is required for the asymptotic normality of S^2 (see Theorem 1). For these reasons, as shown in Polisicchio (2006), when the parent distribution exhibits fat tails, it would be more appropriate to measure the variability using Δ or δ because the properties of their estimators become better than those of S^2 as the weight of the tails increases[2]. Moreover, as for the asymptotic normality of S^2, the CI for the Sharpe Ratio here considered is based on the theoretical assumption of existence of the fourth moment of the parent distribution. Indeed, its coverage accuracy dramatically worsens when sampling from the Student's t with 3 df, which does not possess moments of order greater than 2. This is a potential limitation of the Sharpe Ratio since empirical evidence suggests that the distribution of the returns of many financial assets may have an infinite fourth moment (especially when high frequency returns are considered as shown in Gençay *et al.*, 2001). On the contrary, the CIs for the MAD Ratio and for the MD Ratio (which are based on the less stringent theoretical assumption of existence of the second moment) have a good coverage starting from the sample size of 400 when sampling from the Student's t with 3 df. Another interesting observation is that the actual coverage prob-

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(2) Furthermore, as shown in Polisicchio (2006) and in Johnson, Kotz and Balakrishnan (1995a, ch. 13, p. 136), even if we sample from a Normal distribution in order to estimate σ, the estimators based on $\hat{\delta}$ and $\hat{\Delta}$ given by

$$\hat{\sigma}_{\Delta} = \hat{\Delta}\frac{\sqrt{\pi}}{2} \qquad \textit{and} \qquad \hat{\sigma}_{\delta} = \hat{\delta}\sqrt{\frac{\pi}{2}} \ ,$$

have an efficiency very close to that of S^2 which is, indeed, based on the conjoint sufficient statistics $(\sum_{i=1}^{n} X_i \ ; \ \sum_{i=1}^{n} X_i^2)$ for $(\mu \ ; \ \sigma^2)$.

ability of the CI for the Sharpe Ratio decreases significantly when the true value of the ratio increases, while the coverage probability of the CIs for the MAD Ratio and for the MD Ratio is more stable with respect to variation of the true value of the corresponding ratio. In general, with the exception of the Student's *t* with 3 df, a sample size of 200 is sufficient to reach a good coverage accuracy of the CIs for the MAD Ratio and for the MD Ratio in all the scenarios considered. Concerning the confidence interval for the Sharpe Ratio, a sample size of 400 is still inadequate when sampling from the Skew *t* distribution with negative skewness, which is, indeed, of practical interest.

Another aspect we investigate under the assumption of i.i.d. returns, is the bias of the natural estimators. In particular, we approximate the bias of the three estimators and we introduce approximately unbiased estimators for the three indices. In the simulation study we compare the efficiency of the natural estimators with that of the approximately unbiased ones. We obtain that the natural estimators are, in general, more efficient than the the bias-corrected estimators.

ACKNOWLEDGEMENTS

Lucio De Capitani has done this work during a research fellowship co-funded by ''Dote Ricercatori'', FSE Lombardia.

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