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THE REORDERING VARIATES IN THE DECOMPOSITION BY SOURCES OF INEQUALITY INDEXES

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SUMMARY

Let $X_1, X_2, \ldots, X_c$ be c variates (income sources) observable on each of the n units of a finite population, and Y (total income) be the sum $\sum_{j=1}^{c} X_j$. The n values of Y arranged in non decreasing order are $y_{(1)} \leq \ldots \leq y_{(i)} \leq \ldots \leq y_{(n)}$; $y_{(i)} = \sum_{j=1}^{c} x_{ij}$, where $x_{i1}, \ldots, x_{ij}, \ldots, x_{ic}$ are the values taken by the c variates X_j on the population unit corresponding to $y_{(i)}$. The ordered variate $X_{(j)}, (j = 1, \ldots, c)$, takes on the ordered values $x_{(1j)} \leq \ldots \leq x_{(ij)} \leq \ldots \leq x_{(nj)}$ and $Y^* = \sum_{j=1}^{c} X_{(j)}$ is the total income in the hypothesis of uniform cograduation among the c sources. Zenga et al (2012) have recently proposed a decomposition by sources of Zenga's (2007) point $I_i(Y)$ and synthetic $I(Y)$ inequality indexes of the total Y, as well as a decomposition of the point $I_i(Y^*)$ and synthetic $I(Y^*)$ indexes of Y^*. The decompositions of the two point measures are: $I_i(Y) = \sum_{j=1}^{c} B_i(X_j, Y)$ and $I_i(Y^*) = \sum_{j=1}^{c} B_i(X_{(j)}, Y^*)$, where $B_i(X_j, Y)$ and $B_i(X_{(j)}, Y^*)$ are the contributions of X_j to $I_i(Y)$ and of $X_{(j)}$ to $I_i(Y^*)$. $B(X_j, Y)$ and $B(X_{(j)}, Y^*)$ are the contributions to the corresponding synthetic indexes. The decompositions proposed in this paper are ''finer'' than those given in Zenga, Radaelli and Zenga (2012). The results are obtained using the reordering variates $R_j = X_{(j)} - X_j$ which assume the values: $r_{ij} = x_{(ij)} - x_{ij}$. In particular: $B_i(X_{(j)}, Y^*) = B_i(X_j, Y^*) + B_i(R_j, Y^*)$, where $B_i(X_j, Y^*)$ and $B_i(R_j, Y^*)$ are the contributions of X_j and R_j to the point index $I_i(Y^*)$. In the same way $B_i(X_j, Y) = B_i(X_{(j)}, Y) - B_i(R_j, Y)$. The point index $I_i(X_{(j)})$ of $X_{(j)}$ is decomposed in the sum $B_i(X_j, X_{(j)}) + B_i(R_j, X_{(j)})$, where $B_i(X_j, X_{(j)})$ and $B_i(R_j, X_{(j)})$ are the corresponding contributions of X_j and R_j. Moreover, $B_i(X_j, Y) = \{I_i(X_{(j)}) - B_i(R_j, X_{(j)})\} \cdot {}_j\overset{+}{\gamma}_i \cdot \delta_i$, where ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$, and $\delta_i = \frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}$; $\overset{+}{M}_i(Y)$, $\overset{+}{M}_i(Y^\star)$ and $\overset{+}{M}_i(X_{(j)})$ are the upper means of Y, $Y^*$and $X_{(j)}$, respectively. This relation is also ''extended'' to $B(X_j, X_{(j)})$. These ''finer'' decompositions of $I(Y)$are: compared whit three decompositions by sources of the Gini index $G(Y)$, and applied to the 2008 Bank of Italy sample survey on Household Income and Wealth.

Keywords: *Reordering Variate, Income Inequality, Decomposition by Sources, Point Inequality, Uniform Cograduation.*

1. INTRODUCTION

Let $X_1, \ldots, X_j, \ldots, X_c$ be c variates (income sources) observable on each of the n units of a finite population and let Y be the sum $\sum_{j=1}^{c} X_j$ (total income). Let $y_{(1)} \leq \ldots \leq y_{(i)} \leq \ldots \leq y_{(n)}$ be the n values of Y arranged in non-decreasing order: $y_{(i)} = \sum_{j=1}^{c} x_{ij}$, where $x_{i1}, \ldots, x_{ij}, \ldots, x_{ic}$ are the values taken by the c variates X_j on the population unit corresponding to $y_{(i)}$. Let $X_{(j)}$ be the variate j whose n values

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are arranged in non-decreasing order. In other words, the ordered variate $X_{(j)}, (j = 1, ..., c)$, takes on the ordered values $x_{(1j)} \leq ... \leq x_{(ij)} \leq ... \leq x_{(nj)}$. Let $Y^* = \sum_{j=1}^{c} X_{(j)}$ be the total income in the hypothesis of uniform cograduation among the c sources. Zenga, Radaelli and Zenga (2012) have recently proposed a decomposition by sources of the Zenga (2007) point $I_i(Y)$ and synthetic $I(Y) = \frac{1}{n}\sum_i I_i(Y)$ inequality indexes of the total Y, as well as a decomposition of the point $I_i(Y^*)$ and synthetic $I(Y^*) = \frac{1}{n}\sum_i I_i(Y^*)$ indexes of Y^*. This paper proposes ''finer'' decompositions than those given in Zenga *et al.* (2012) .

The paper is organized as follows. The next section introduces the point and synthetic Zenga inequality indexes of the total incomes Y and Y^*, and of the ordered components $X_{(j)}$. In Section 3 the contributions $B_i(X_j, Y)$ and $B(X_j, Y)$ of X_j to $I_i(Y)$ and $I(Y)$ are introduced. Section 4 defines the contributions $B_i(X_{(j)}, Y^*)$ and $B(X_{(j)}, Y^*)$ of $X_{(j)}$ to $I_i(Y^*)$ and $I(Y^*)$. In Section 5 is introduced the identity $X_{(j)} = X_j + R_j$, where $R_j = X_{(j)} - X_j$ is the reordering variate. Then, it is shown that $B_i(X_{(j)}, Y^*) = B_i(X_j, Y^*) + B_i(R_j, Y^*)$ and that $B_i(X_{(j)}, Y) = B_i(X_j, Y) + B_i(R_j, Y)$, where $B_i(X_j, Y^*)$ and $B_i(R_j, Y^*)$ are respectively the contributions of X_j and R_j to $I_i(Y^*)$ and where $B_i(X_{(j)}, Y)$ and $B_i(R_j, Y)$ are respectively the contributions of $X_{(j)}$ and R_j to $I_i(Y)$. Moreover, in this section it is proved that $B_i(R_j, Y^*) \geq 0$ and that $B_i(R_j, Y) \geq 0$, $\forall(i,j)$. Section 5.1 illustrates an additive $c \times 2$ bivariate decomposition of the point and synthetic inequality indexes. In Section 6 the point index $I_i(X_{(j)})$ of $X_{(j)}$ is decomposed in the sum $B_i(X_j, X_{(j)}) + B_i(R_j, X_{(j)})$, where $B_i(X_j, X_{(j)})$ and $B_i(R_j, X_{(j)})$ are the contributions of X_j and R_j to $I_i(X_{(j)})$. Then, it is shown that $B_i(X_j, Y) = \{I_i(X_{(j)}) - B_i(R_j, X_{(j)})\} \cdot {}_j\overset{+}{\gamma}_i \cdot \delta_i$, where: ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$, $\delta_i = \frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}$, and $\overset{+}{M}_i(Y)$, $\overset{+}{M}_i(Y^\star)$ and $\overset{+}{M}_i(X_{(j)})$ are the upper means of Y, $Y^*$and $X_{(j)}$, respectively. This relation is also ''extended'' to $B(X_j, Y)$. In Section 7 the decomposition by sources of Gini and Zenga indexes are compared. Finally, in Section 8 the new decompositions by sources are applied to Italian household income collected by the 2008 Survey on Household Income and Wealth of the Bank of Italy. Section 9 is devoted to conclusions and final remarks.

2. DEFINITIONS AND NOTATION

Let $X_1, X_2, ..., X_c$ be c variates (income sources) observable on each of the n units (individuals or households) of a finite population and let Y be the sum $\sum_{j=1}^{c} X_j$ (total income).

Let

$$0 \leq y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)} > 0 \tag{1}$$

be the n values of Y arranged in non-decreasing order. The values of the c income sources referring to a single population unit may be reported in a row of a matrix and the rows may be ordered according to the values of the variate Y. In this way we get, (Table 1), the ''Data matrix (DM) with rows ordered according to $y_{(i)}$'',

TABLE 1. - *Data matric (DM)*

X_1	$\cdots$	X_j	$\cdots$	X_c	Y
x_{11}	$\cdots$	x_{1j}	$\cdots$	x_{1c}	$y_{(1)}$
$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{i1}	$\cdots$	x_{ij}	$\cdots$	x_{ic}	$y_{(i)}$
$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{n1}	$\cdots$	x_{nj}	$\cdots$	x_{nc}	$y_{(n)}$
${}_1T$	$\cdots$	${}_jT$	$\cdots$	${}_cT$	T

where:

$$x_{i1} + \cdots x_{ij} + \cdots + x_{ic} = y_{(i)}, \ i = 1, 2, \ldots, n, \tag{2}$$

and $x_{i1}, ..., x_{ij}, ..., x_{ic}$ are the values taken by the c variates X_j, $(j = 1, \ldots, c)$, on the population unit corresponding to $y_{(i)}$.

Let:

- ${}_jT = \sum_{i=1}^{n} x_{ij}$ be the sum of the n values of the columns j $(j = 1, \ldots, c)$;
- $T = \sum_{j=1}^{c} {}_jT = \sum_{i=1}^{n} y_{(i)}$ be the overall income of the whole population;
- $M = \frac{1}{n} \cdot T$ and ${}_jM = \frac{1}{n} \cdot {}_jT$ be the means of Y and X_j, respectively;
- ${}_j\gamma = \frac{{}_jM}{M} = \frac{{}_jT}{T}$ be the share of X_j on T, (on M), $\sum_{j=1}^{c} {}_j\gamma = 1$.

Now, we will define with reference to the total income Y, Zenga's (2007) inequality index. Let:

$Q_i(Y) = \sum_{t=1}^{i} y_{(t)}$, $\quad (i = 1, 2, \ldots, n)$, be the overall income of the i poorest population units; note that $Q_n(Y) = T$.

At each value $y_{(i)}$ we can split the population into two disjoint subgroups, getting:

- a *lower group* that contains the i poorest population units with total incomes $\left(y_{(1)}, \ldots, y_{(i)}\right)$ and *lower mean*

$$\overset{-}{M}_i(Y) = \frac{1}{i} \cdot Q_i(Y), \tag{3}$$

- and an *upper group* that contains the remaining (richer) part of the population with total incomes $\left(y_{(i+1)}, \ldots, y_{(n)}\right)$ and *upper mean*

$$\overset{+}{M}_i(Y) = \begin{cases} \dfrac{T - Q_i(Y)}{n - i}, & (i = 1, 2, \ldots, n - 1) \\ y_{(n)}, & (i = n). \end{cases} \tag{4}$$

Zenga (2007) proposed to measure inequality between the lower and the upper group of Y by means of the ratios

$$I_i(Y) = \frac{\overset{+}{M}_i(Y) - \overset{-}{M}_i(Y)}{\overset{+}{M}_i(Y)}, \quad (i = 1, 2, \ldots, n). \tag{5}$$

Note that $I_i(Y)$ is a point inequality measure that gives the relative variation of $\overset{-}{M}_i(Y)$ w.r.t. $\overset{+}{M}_i(Y)$. The synthetic measure of inequality is then given by

$$I(Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(Y) \tag{6}$$

From the data matrix (DM) we may obtain another matrix arranging each column in non-decreasing order. In this matrix:

$$x_{(1j)} \leq \ldots \leq x_{(ij)} \leq \ldots \leq x_{(nj)}, \quad j = 1, 2, \ldots, c,$$

and we will refer to it as the ‘‘Reordered Data matrix (RDM)’’ (Table 2). Note that while each row of the data matrix (DM) refers to a single population unit, this is not generally true for the rows in the reordered data matrix (RDM). Adding up the rows in the reordered data matrix we obtain the theoretical values

$$y^{\star}_{(i)} = x_{(i1)} + \ldots + x_{(ij)} + \ldots + x_{(ic)}, \quad i = 1, 2, \ldots, n. \tag{7}$$

TABLE 2. - *Reordered Data matrix (RDM)*

$X_{(1)}$	$\cdots$	$X_{(j)}$	$\cdots$	X_c	Y^*
$x_{(11)}$	$\cdots$	$x_{(1j)}$	$\cdots$	$x_{(1c)}$	$y^*_{(1)}$
$\vdots$		$\vdots$		$\vdots$	$\vdots$
$x_{(i1)}$	$\cdots$	$x_{(ij)}$	$\cdots$	$x_{(ic)}$	$y^*_{(i)}$
$\vdots$		$\vdots$		$\vdots$	$\vdots$
$x_{(n1)}$	$\cdots$	$x_{(nj)}$	$\cdots$	$x_{(nc)}$	$y^*_{(n)}$
${}_1T$	$\cdots$	${}_jT$	$\cdots$	${}_cT$	T

Let :

- $X_{(j)}$ be the variate that assumes the n ordered values $\left(x_{(1j)}, \ldots, x_{(nj)}\right)$ of column j, $(j = 1, \ldots, c)$, of the Reordered data matrix ;
- Y^* be the variate that assumes the n ordered values $\left(y^{\star}_{(1)}, \ldots, y^{\star}_{(i)}, \ldots, y^{\star}_{(n)}\right)$, where $y^{\star}_{(i)} = \sum_{j=1}^{c} x_{(ij)}$. Obviously, Y^* is the sum $\sum_{j=1}^{c} X_{(j)}$ (total income in the reordered data matrix);
- $Q_i(Y^*) = \sum_{t=1}^{i} y^*_{(t)}, \quad (i = 1, 2, \ldots, n)$.

Let:

$$\overset{-}{M}_i(Y^*) = \frac{1}{i} \cdot Q_i(Y^*), \quad (i = 1, 2, \ldots, n) \tag{8}$$

$$\overset{+}{M}_i(Y^*) = \begin{cases} \dfrac{T - Q_i(Y^*)}{n - i}, & (i = 1, 2, \ldots, n-1) \\ y^*_{(n)}, & (i = n). \end{cases} \tag{9}$$

Zenga's point index for Y^* is furnished by

$$I_i(Y^*) = \frac{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)}{\overset{+}{M}_i(Y^*)}, \quad (i = 1, 2, \ldots, n) \tag{10}$$

It is easy to show that $Q_i(Y^*) \leq Q_i(Y), \quad (i = 1, 2, \ldots, n)$. Therefore:

$$\overset{-}{M}_i(Y^*) \leq \overset{-}{M}_i(Y), \overset{+}{M}_i(Y^*) \geq \overset{+}{M}_i(Y), I_i(Y^*) \geq I_i(Y), (i = 1, 2, \ldots, n). \tag{11}$$

Thus,

$$I(Y^*) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(Y^*) \geq \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(Y) = I(Y), \tag{12}$$

with equality only in the case that among the c sources there is *uniform cogradua-tion*:

$$x_{ij} = x_{(ij)}, \forall (i,j).$$

Let:

$$Q_i\left(X_{(j)}\right) = \textstyle\sum_{t=1}^{i} x_{(tj)}, \quad (i = 1, 2, \ldots, n);$$

$$\overset{-}{M}_i\left(X_{(j)}\right) = \frac{1}{i} \cdot Q_i\left(X_{(j)}\right), \quad (i = 1, 2, \ldots, n); \tag{13}$$

$$\overset{+}{M}_i\left(X_{(j)}\right) = \begin{cases} \dfrac{{}_jT - Q_i\left(X_{(j)}\right)}{n - i,} & (i = 1, 2, \ldots, n-1) \\ x_{(nj)}, & (i = n). \end{cases} \tag{14}$$

Zenga's indexes for $X_{(j)}$ are given by:

$$I_i\left(X_{(j)}\right) = \frac{\overset{+}{M}_i\left(X_{(j)}\right) - \overset{-}{M}_i\left(X_{(j)}\right)}{\overset{+}{M}_i\left(X_{(j)}\right)}, \quad (i = 1, 2, \ldots, n); \tag{15}$$

$$I\left(X_{(j)}\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i\left(X_{(j)}\right) \tag{16}$$

EXAMPLE 1

Tables 3, 4 and 5 illustrate Zenga's point and synthetic indexes of Y, Y^* and $X_{(j)}$.

TABLE 3. - *Data matrix and reordered data matrix*

i	x_{i1}	x_{i2}	x_{i3}	$y_{(i)}$	$x_{(i1)}$	$x_{(i2)}$	$x_{(i3)}$	$y^*_{(i)}$
1	7	6	5	18	3	2	5	10
2	6	11	5	22	6	6	5	17
3	3	15	10	28	7	6	10	23
4	14	6	18	38	14	11	12	37
5	30	2	12	44	30	15	18	63
${}_jT$	60	40	50	150	60	40	50	150
${}_j\gamma$	0.4	$0.2\bar{6}$	$0.3\bar{3}$	1.00	0.4	$0.2\bar{6}$	$0.3\bar{3}$	1.00

TABLE 4. - *Points and synthetic inequality indexes of Y and Y^**

i	$Q_i(Y)$	$\bar{M}_i(Y)$	$\overset{+}{M}_i(Y)$	$I_i(Y)$	$Q_i(Y^*)$	$\bar{M}_i(Y^*)$	$\overset{+}{M}_i(Y^*)$	$I_i(Y^*)$
1	18	18	33	0.455	10	10	35	0.714
2	40	20	$36.\bar{6}$	0.455	27	13.5	41	0.671
3	68	$22.\bar{6}$	41	0.447	50	$16.6\bar{6}$	50	0.667
4	106	26.5	44	0.398	87	21.75	63	0.655
5	150	30	44	0.318	150	30	63	0.524
				0.411 $= I(Y)$				0.646 $= I(Y^*)$

TABLE 5. - *Points $I_i(X_j)$ and synthetic $I(x_{(j)}$ inequality indexes of $X_{(j)}$, $(j = 1, 2, 3)$*

i	$Q_i(X_{(1)})$	$I_i(X_{(1)})$	$Q_i(X_{(2)})$	$I_i(X_{(2)})$	$Q_i(X_{(3)})$	$I_i(X_{(3)})$
1	3	0.789	2	0.789	5	0.555
2	9	0.735	8	0.625	10	0.625
3	16	0.758	14	0.641	20	0.555
4	30	0.75	25	0.583	32	0.555
5	60	0.6	40	0.467	50	0.444
		0.727 $= I(X_{(1)})$		0.621 $= I(X_{(2)})$		0.547 $= I(X_{(3)})$

3. ADDITIVE DECOMPOSITION OF THE INDEX $I_i(Y)$ AND $I(Y)$

Recently Zenga, *et al.* (2012) obtained the additive contributions of each variate X_j to the point and synthetic inequality indexes $I_i(Y)$ and $I(Y)$. To obtain the result, let us consider the n values $(x_{1j}, \ldots, x_{nj})$ of income source X_j in column j of the data matrix (Table 1). Let:

$$Q_i(X_j) = \sum_{t=1}^{i} x_{tj}, \quad (i = 1, 2, \ldots, n);$$

$$\bar{M}_i(X_j) = \frac{1}{i} \cdot Q_i(X_j), \quad (i = 1, 2, \ldots, n), \tag{17}$$

be the mean of the variate X_j in the lower group;

$$\overset{+}{M}_i(X_j) = \begin{cases} \dfrac{{}_jT - Q_i(X_j)}{n - i}, & (i = 1, 2, \ldots, n-1) \\ x_{nj}, & (i = n), \end{cases} \tag{18}$$

be the mean of the variate X_j in the upper group.

Now, from (5) and the relations $\overset{-}{M}_i(Y) = \sum_{j=1}^{c} \overset{-}{M}_i(X_j)$ and $\overset{+}{M}_i(Y) = \sum_{j=1}^{c} \overset{+}{M}_i(X_j)$, we obtain:

$$I_i(Y) = \sum_{j=1}^{c} B_i(X_j, Y), \tag{19}$$

where: $B_i(X_j, Y) = \frac{\overset{+}{M}_i(X_j) - \overset{-}{M}_i(X_j)}{\overset{+}{M}_i(Y)}$ is the contribution of X_j to $I_i(Y)$. From (6) and (19) we obtain:

$$I(Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} \sum_{j=1}^{c} B_i(X_j, Y) = \sum_{j=1}^{c} B(X_j, Y), \tag{20}$$

where: $B(X_j, Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} B_i(X_j, Y)$, is the contribution of X_j to $I(Y)$. The relative contribution of X_j to $I_i(Y)$ is given by

$$\beta_i(X_j, Y) = \frac{B_i(X_j, Y)}{I_i(Y)} = \frac{\overset{+}{M}_i(X_j) - \overset{-}{M}_i(X_j)}{\overset{+}{M}_i(Y) - \overset{-}{M}_i(Y)}, \quad (I_i(Y) > 0), \tag{21}$$

obviously $\sum_{j=1}^{c} \beta_i(X_j, Y) = 1$. The relative contribution of X_j to $I(Y)$ is given by:

$$\beta(X_j, Y) = \frac{B(X_j, Y)}{I(Y)} = \sum_i \beta_i(X_j, Y) \cdot \frac{I_i(Y)}{n \cdot I(Y)}, \quad (I(Y) > 0). \tag{22}$$

In Table 6 the computations to obtain the contributions $B_i(X_j, Y)$ and $B(X_j, Y)$ are illustrated.

TABLE 6. - *Computations to obtain the contributions $B_i(X_j, Y)$ and $B(X_j, Y)$*

i	$Q_i(X_1)$	$B_i(X_1, Y)$	$Q_i(X_2)$	$B_i(X_2, Y)$	$Q_i(X_3)$	$B_i(X_3, Y)$
1	7	0.189	6	0.076	5	0.189
2	13	0.250	17	−0.023	10	0.227
3	16	0.407	32	−0.163	20	0.203
4	30	0.511	38	−0.17	38	0.057
5	60	0.409	40	−0.136	50	0.045
		0.353 $= B(X_1, Y)$		−0.083 $= B(X_2, Y)$		0.144 $= B(X_3, Y)$

4. ADDITIVE DECOMPOSITION BY SOURCES OF THE INEQUALITY INDEXES $I_i(Y^*)$ AND $I(Y^*)$ IN THE CASE OF REORDERED DATA MATRIX

From (10) and the relations $\overset{-}{M}_i(Y^*) = \sum_{j=1}^{c} \overset{-}{M}_i\big(X_{(j)}\big)$ and $\overset{+}{M}_i(Y^*) = \sum_{j=1}^{c} \overset{+}{M}_i\big(X_{(j)}\big)$ we get:

$$I_i(Y^*) = \sum_{j=1}^{c} B_i\big(X_{(j)}, Y^*\big), \tag{23}$$

where $B_i\big(X_{(j)}, Y^*\big) = \frac{\overset{+}{M}_i(X_{(j)}) - \overset{-}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$, is the contribution of $X_{(j)}$ to $I_i(Y^*)$. Averaging (over i) in (23) we obtain:

$$I(Y^*) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{c} B_i\big(X_{(j)}, Y^*\big) = \sum_{j=1}^{c} B\big(X_{(j)}, Y^*\big), \tag{24}$$

where $B\big(X_{(j)}, Y^*\big) = \frac{1}{n} \sum_{i=1}^{n} B_i\big(X_{(j)}, Y^*\big)$, is the contribution of $X_{(j)}$ to $I(Y^*)$. The relative contribution of $X_{(j)}$ to $I_i(Y^*)$ is given by:

$$\beta_i\big(X_{(j)}, Y^*\big) = \frac{B_i\big(X_{(j)}, Y^*\big)}{I_i(Y^*)} = \frac{\overset{+}{M}_i\big(X_{(j)}\big) - \overset{-}{M}_i\big(X_{(j)}\big)}{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)}, \tag{25}$$

obviously $\sum_{j=1}^{c} \beta_i\big(X_{(j)}, Y^*\big) = 1$. The relative contribution of $X_{(j)}$ to $I(Y^*)$ is given by:

$$\beta\big(X_{(j)}, Y^*\big) = \frac{B\big(X_{(j)}, Y^*\big)}{I(Y^*)} = \sum_{i=1}^{n} \beta_i\big(X_{(j)}, Y^*\big) \cdot \frac{I_i(Y^*)}{n \cdot I(Y^*)}. \tag{26}$$

The contribution $B_i\big(X_{(j)}, Y^*\big)$ can be written as follows:

$$B_i\big(X_{(j)}, Y^*\big) = \frac{\overset{+}{M}_i\big(X_{(j)}\big) - \overset{-}{M}_i\big(X_{(j)}\big)}{\overset{+}{M}_i(Y^*)} \cdot \frac{\overset{+}{M}_i\big(X_{(j)}\big)}{\overset{+}{M}_i\big(X_{(j)}\big)} = I_i\big(X_{(j)}\big) \cdot {}_j\overset{+}{\gamma}_i, \tag{27}$$

where

$${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i\big(X_{(j)}\big)}{\overset{+}{M}_i(Y^*)}, \tag{28}$$

is the share of $\overset{+}{M}_i\big(X_{(j)}\big)$ on $\overset{+}{M}_i(Y^*)$, $\sum_{j=1}^{c} {}_j\overset{+}{\gamma}_i = 1$. Now, from (23), (27) and (24) we have:

$$\begin{cases} I_i(Y^*) & = \sum_{j=1}^{c} I_i\big(X_{(j)}\big) \cdot {}_j\overset{+}{\gamma}_i \\ I(Y^*) & = \sum_{j=1}^{c} \frac{1}{n} \sum_{i=1}^{n} I_i\big(X_{(j)}\big) \cdot {}_j\overset{+}{\gamma}_i \end{cases} \tag{29}$$

Note that it is possible to extend the relation (27) between $B_i(X_{(j)}, Y^*)$ and $I_i(X_{(j)})$ to the following relation between $B(X_{(j)}, Y^*)$ and $I(X_{(j)})$:

$$B(X_{(j)}, Y^*) = I(X_{(j)}) \cdot_j \overset{*}{\gamma}, \text{ where } {}_j\overset{*}{\gamma} = \sum_{i=1}^{n} {}_j\overset{+}{\gamma}_i \cdot \frac{I_i(X_{(j)})}{n \cdot I(X_{(j)})}.$$

Finally, we have: $I(Y^*) = \sum_{j=1}^{c} I(X_j) \cdot {}_j\overset{*}{\gamma}$.

The contributions $B_i(X_{(j)}, Y^*)$ and $B(X_{(j)}, Y^*)$, and the shares ${}_j\overset{+}{\gamma}_i$ and ${}_j\overset{*}{\gamma}$, are reported in Table 7.

TABLE 7. - *Contributions $B_i(X_{(j)}, Y^*)$ and $B(X_{(j)}, Y^*)$, shares ${}_j\overset{+}{\gamma}_i$ and ${}_j\overset{*}{\gamma}$*

i	$B_i(X_{(1)}, Y^*)$	$B_i(X_{(2)}, Y^*)$	$B_i(X_{(3)}, Y^*)$	$I_i(Y^*)$	${}_1\overset{+}{\gamma}_i$	${}_2\overset{+}{\gamma}_i$	${}_3\overset{+}{\gamma}_i$	
1	0.321	0.214	0.179	0.714	0.407	0.274	0.321	1
2	0.305	0.163	0.203	0.671	0.415	0.260	0.325	1
3	0.333	0.167	0.167	0.666	0.44	0.260	0.3	1
4	0.357	0.139	0.159	0.655	0.476	0.238	0.286	1
5	0.286	0.111	0.127	0.524	0.476	0.238	0.286	
$B(X_{(j)}, Y^*)$	0.32	0.159	0.167	$I(Y^*) =$ 0.646				
$I(X_{(j)})$	0.727	0.621	0.547		0.440	0.256	0.305	${}_j\overset{*}{\gamma}$

5. THE REORDERING VARIATE R_j AND THE DECOMPOSITIONS IN TWO TERMS OF $B_i(X_{(j)}, Y^*)$ AND $B_i(X_j, Y)$

Let $R_j = X_{(j)} - X_j$ be the difference variate between $X_{(j)}$ and X_j. The variate R_j takes the values: $r_{ij} = x_{(ij)} - x_{ij}, \quad (i = 1, 2, \ldots, n)$. In other words, $X_{(j)} = X_j + R_j$ and

$$x_{(ij)} = x_{ij} + r_{ij}, \quad (i = 1, 2, \ldots, n). \tag{30}$$

R_j can be called reordering variate, since for each $i = 1, \ldots, n, r_{ij}$ is the value to add to x_{ij} to obtain the ordered value $x_{(ij)}$. The lower and the upper means of R_j are:

$$\overset{-}{M}_i(R_j) = \tfrac{1}{i}\sum_{t=1}^{i} r_{tj}, (i = 1, \ldots, n) \text{ and}$$

$$\overset{+}{M}_i(R_j) = \begin{cases} \frac{1}{n-i}\sum_{t=i+1}^{n} r_{tj}, & (i = 1, \ldots, n-1) \\ r_{nj}, & (i = n). \end{cases}$$

Obviously $\overset{-}{M}_i(X_{(j)}) = \overset{-}{M}_i(X_j) + \overset{-}{M}_i(R_j)$ and $\overset{+}{M}_i(X_{(j)}) = \overset{+}{M}_i(X_j) + \overset{+}{M}_i(R_j)$. Consequently, from the definition of $B_i(X_{(j)}, Y^*)$ we obtain:

$$B_i(X_{(j)}, Y^*) = \frac{\overset{+}{M}_i(X_{(j)}) - \overset{-}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)} = B_i(X_j, Y^*) + B_i(R_j, Y^*), \tag{31}$$

where $B_i(X_j, Y^*) = \frac{\overset{+}{M}_i(X_j) - \overset{-}{M}_i(X_j)}{\overset{+}{M}_i(Y^*)}$ and $B_i(R_j, Y^*) = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(Y^*)}$ are respectively the contributions of X_j and R_j to $B_i(X_{(j)}, Y^*)$. Averaging (over i) we obtain the decomposition:

$$B(X_{(j)}, Y^*) = B(X_j, Y^*) + B(R_j, Y^*) \tag{32}$$

where, $B(X_j, Y^*) = \frac{1}{n}\sum_{i=1}^{n} B_i(X_j, Y^*)$, and $B(R_j, Y^*) = \frac{1}{n}\sum_{i=1}^{n} B_i(R_j, Y^*)$. For the interpretation of (31) and (32) it is useful the following Lemma 1.

LEMMA 1
$\left\{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)\right\} \geq 0, \forall(i,j).$

PROOF:
It is easy to show that $Q_i(X_{(j)}) \leq Q_i(X_j), \ \forall(i)$. Thus, $\overset{-}{M}_i(X_{(j)}) \leq \overset{-}{M}_i(X_j)$ and $\overset{+}{M}_i(X_{(j)}) \geq \overset{+}{M}_i(X_j)$. Consequently: $\overset{-}{M}_i(R_j) \leq 0$ and $\overset{+}{M}_i(R_j) \geq 0$. Then,

$$\left\{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)\right\} \geq 0, \forall(i,j).$$

From Lemma 1 it derives that:

$$B_i(R_j, Y^*) \geq 0, \ \forall(i,j), \ \text{ and that } \ B(R_j, Y^*) \geq 0, \ \forall(j). \tag{33}$$

Relation (33) informs that the contribution $B(R_j, Y^*)$ to $B(X_{(j)}, Y^*)$ resulting from the reordering of X_j is greater or equal to 0, with equality only in the case: $x_{1j} \leq x_{2j} \leq \cdots \leq x_{nj}$.

Table 8 reports the reordering values $r_{ij} = x_{(ij)} - x_{ij}$, Table 9 reports the lower and the upper means of the reordering variates, and Table 10 reports the contributions $B_i(R_j, Y^*), B(R_j, Y^*)$ and $\sum_j B_i(R_j, Y^*)$.

TABLE 8. - *Reordering values* $r_{ij} = x_{(ij)} - x_{ij}$

i	r_{i1}	r_{i2}	r_{i3}	$y^*_{(i)} - y_{(i)}$
1	-4	-4	0.0	-8
2	0	-5	0.0	-5
3	$+4$	-9	0.0	-5
4	0	$+5$	-6	-1
5	0	$+13$	$+6$	$+19$
	0.0	0.0	0.0	0.0

TABLE 9. - *Lower and upper means of the reordering variates* R_j

$Q_i(R_1)$	$\overset{-}{M}_i(R_1)$	$\overset{+}{M}_i(R_1)$	$Q_i(R_2)$	$\overset{-}{M}_i(R_2)$	$\overset{+}{M}_i(R_2)$	$Q_i(R_3)$	$\overset{-}{M}_i(R_3)$	$\overset{+}{M}_i(R_3)$
−4	−4	1	−4	−4	1	0	0	0
−4	−2	1.33	−9	−4.5	3	0	0	0
0	0	0	−18	−6	9	0	0	0
0	0	0	−13	−3.25	13	−6	−1.5	6
0	0	0	0	0	13	0	0	6

TABLE 10. - *Contributions* $B_i(R_j, Y^*), B(R_j, Y^*)$ *and* $\sum_j B_i(R_j, Y^*) = B_i(R, Y^*)$

i	$B_i(R_1, Y^*)$	$B_i(R_2, Y^*)$	$B_i(R_3, Y^*)$	$B_i(R, Y^*)$
1	0.143	0.143	0.000	0.286
2	0.081	0.183	0.000	0.264
3	0.000	0.300	0.000	0.300
4	0.000	0.258	0.119	0.377
5	0.000	0.206	0.095	0.301
	0.045	0.218	0.043	0.306
	$B(R_1, Y^*)$	$B(R_2, Y^*)$	$B(R_3, Y^*)$	$B(R, Y^*)$

Now we will obtain for $B_i(X_j, Y)$, a two subtractive terms decomposition. From the relations $\overset{+}{M}_i(X_j) = \overset{+}{M}_i(X_{(j)}) - \overset{+}{M}_i(R_j)$ and $\overset{-}{M}_i(X_j) = \overset{-}{M}_i(X_{(j)}) - \overset{-}{M}_i(R_j)$, we obtain:

$$B_i(X_j, Y) = B_i(X_{(j)}, Y) - B_i(R_j, Y), \tag{34}$$

where $B_i(X_{(j)}, Y) = \frac{\overset{+}{M}_i(X_{(j)}) - \overset{-}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y)}$ and $B_i(R_j, Y) = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(Y)}$ are respectively the contributions of $X_{(j)}$ and R_j to $B_i(X_j, Y)$. Averaging (over i) we obtain:

$$B(X_j, Y) = B(X_{(j)}, Y) - B(R_j, Y), \text{ where} \tag{35}$$

$$B(X_{(j)}, Y) = \frac{1}{n}\sum_{i=1}^{n} B_i(X_{(j)}, Y), \text{ and } \quad B(R_j, Y) = \frac{1}{n}\sum_{i=1}^{n} B_i(R_j, Y).$$

Notice that from Lemma 1: $B_i(R_j, Y) \geq 0, \ \forall(i,j),$ *and* $B(R_j, Y) \geq 0, \ \forall(j)$.

5.1 *Additive* $c \times 2$ *bivariate decomposition of* $I_i(Y^*)$, $I(Y^*)$, $I_i(Y)$ *and* $I(Y)$

The variate Y^* can be represented by the sums:

$$\begin{aligned} Y^* &= \{X_{(1)} + \cdots + X_{(j)} + \cdots + X_{(c)}\}, \\ &= \{(X_1 + R_1) + \cdots + (X_j + R_j) + \cdots + (X_c + R_c)\}, \\ &= \{Y + R\}, \end{aligned} \tag{36}$$

where $R = \sum_j R_j$ takes the values:

$$r_i = \sum_j r_{ij} = \sum_j \left(x_{(ij)} - x_{ij}\right) = y^*_{(i)} - y_{(i)},\ \forall (i).$$

It is possible to decompose $I_i(Y^*)$ by the components of each sum (36). In particular:

$$I_i(Y^*) = \sum_j B_i\left(X_{(j)}, Y^*\right),$$

$$= \sum_j \left(B_i\left(X_j, Y^*\right) + B_i\left(R_j, Y^*\right)\right), \tag{37}$$

$$= B_i(Y, Y^*) + B_i(R, Y^*), \tag{38}$$

where $B_i\left(X_j, Y^*\right)$ and $B_i\left(R_j, Y^*\right)$ are defined in (31) , and where

$$B_i(Y, Y^*) = \frac{\overset{+}{M}_i(Y) - \overset{-}{M}_i(Y)}{\overset{+}{M}_i(Y^*)} \tag{39}$$

$$B_i(R, Y^*) = \frac{\overset{+}{M}_i(R) - \overset{-}{M}_i(R)}{\overset{+}{M}_i(Y^*)}. \tag{40}$$

Table 11 reports the decompositions (23), (37) and (38) of $I_i(Y^*)$ and the corresponding decompositions of $I(Y^*)$. Tables 12 and 13 illustrate, for the example at hand, the decomposition of $I_1(Y^*)$ and $I(Y^*)$.

TABLE 11. - *Contributions of the variates $X_j, R_j, X_{(j)}, Y$ and R to the indexes $I_i(Y^*)$ and $I(Y^*)$*

1	$B_i(X_1, Y^*)$	$B_i(R_1, Y^*)$	$B_i\left(X_{(1)}, Y^*\right)$	$B(X_1, Y^*)$	$B(R_1, Y^*)$	$B\left(X_{(1)}, Y^*\right)$
$\vdots$						
j	$B_i\left(X_j, Y^*\right)$	$B_i\left(R_j, Y^*\right)$	$B_i\left(X_{(j)}, Y^*\right)$	$B\left(X_j, Y^*\right)$	$B\left(R_j, Y^*\right)$	$B\left(X_{(j)}, Y^*\right)$
$\vdots$						
c	$B_i(X_c, Y^*)$	$B_i(R_c, Y^*)$	$B_i\left(X_{(c)}, Y^*\right)$	$B(X_c, Y^*)$	$B(R_c, Y^*)$	$B\left(X_{(c)}, Y^*\right)$
	$B_i(Y, Y^*)$	$B_i(R, Y^*)$	$I_i(Y^*)$	$B(Y, Y^*)$	$B(R, Y^*)$	$I(Y^*)$

TABLE 12. - *Contributions and relative contributions of the variates $X_j, R_j, X_{(j)}, Y$ and R to the point index $I_1(Y^*)$. Example at hand.*

j	$B_1(X_j, Y^*)$	$B_1(R_j, Y^*)$	$B_1(X_{(j)}, Y^*)$	$\beta_1(X_j, Y^*)$	$\beta_1(R_j, Y^*)$	$\beta_1(X_{(j)}; Y^*)$
1	0.178	0.143	0.321	0.249	0.2	0.449
2	0.071	0.143	0.214	0.1	0.2	0.3
3	0.179	0	0.179	0.251	0	0.251
	0.428	0.286	0.714	0.6	0.4	1
	$B_1(Y, Y^*)$	$B_1(R, Y^*)$	$I_1(Y^*)$	$\beta_1(Y, Y^*)$	$\beta_1(R, Y^*)$	

From Table 12 we can deduce many important informations. For example the relative contributions of Y and of R to the point index $I_1(Y^*)$ are given by $\beta_1(Y, Y^*) = \frac{B_1^*(Y,Y^*)}{I_1(Y^*)} = \frac{0.428}{0.714} = 0.6$, and by $\beta_1(R, Y^*) = \frac{B_1(R,Y^*)}{I_1(Y^*)} = \frac{0.286}{0.714} = 0.4$. From Table 12 we deduce also that the contribution to $I_1(Y^*)$ of X_2 is smaller than the one of R_2. The ratio $\frac{B_1(R_j,Y^*)}{B_1(X_{(j)},Y^*)}$ gives the ''influence'' of the reordering variate R_j on the contribution $B_1(X_{(j)}, Y^*)$: $\frac{B_1(R_1,Y^*)}{B_1(X_{(1)},Y^*)} = 0.445$; $\frac{B_1(R_2,Y^*)}{B_1(X_{(2)},Y^*)} = 0.668$; $\frac{B_1(R_3.Y^*)}{B_1(X_{(3)},Y^*)} = 0.0$.

TABLE 13. - *Contributions and relative contributions of the variates $X_j, R_j, X_{(j)}, Y$ and R to the synthetic index $I(Y^*)$. Example at hand*

j	$B(X_j, Y^*)$	$B(R_j, Y^*)$	$B(X_{(j)}, Y^*)$	$\beta(X_j, Y^*)$	$\beta(R_j, Y^*)$	$\beta(X_{(j)}, Y^*)$
1	0.276	0.044	0.32	0.427	0.068	0.495
2	−0.059	0.218	0.159	−0.091	0.337	0.246
3	0.124	0.043	0.167	0.192	0.067	0.259
	0.340	0.305	0.646	0.528	0.472	1.0
	$B(Y, Y^*)$	$B(R, Y^*)$	$I(Y^*)$	$\beta(Y, Y^*)$	$\beta(R, Y^*)$	

Many important informations are also reported in Table 13: for example we note that it is particularly relevant the contribution of the reordering variate R_2 to the value of $I(Y^*)$.

The variate Y can be represented by the sums:

$$\begin{aligned} Y &= \{X_1 + \cdots + X_j + \cdots + X_c\}, \\ &= \{(X_{(1)} - R_1) + \cdots + (X_{(j)} - R_j) + \cdots + (X_{(c)} - R_c)\}, \\ &= \{Y^* - R\}. \end{aligned} \tag{41}$$

It is possible to decompose $I_i(Y)$ by the components of each sum (41). In particular

$$I_i(Y) = \sum_j B_i\big(X_j, Y\big),$$

$$= \sum_j \Big(B_i\big(X_{(j)}, Y\big) - B_i\big(R_j, Y\big)\Big), \quad (42)$$

$$= B_i(Y^*, Y) - B_i(R, Y), \quad (43)$$

where $B_i\big(X_{(j)}, Y\big)$ and $B_i\big(R_j, Y\big)$ are defined in (34) , and where

$$B_i(Y^*, Y) = \frac{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)}{\overset{+}{M}_i(Y)} \quad (44)$$

$$B_i(R, Y) = \frac{\overset{+}{M}_i(R) - \overset{-}{M}_i(R)}{\overset{+}{M}_i(Y)}. \quad (45)$$

Tables 14 and 15 illustrate, for the example at hand, the bivariate decompositions of $I_1(Y)$ and $I(Y)$.

TABLE 14. - *Contributions and relative contributions of the variates $X_j, R_j, X_{(j)}, Y^*$ and R to the point index $I_1(Y)$. Example at hand*

j	$B_1\big(X_{(j)}, Y\big)$	$-B_1\big(R_j, Y\big)$	$B_1\big(X_j, Y\big)$	$\beta_1\big(X_{(j)}, Y\big)$	$-\beta_1\big(R_j, Y\big)$	$\beta_1\big(X_j, Y\big)$
1	0.341	−0.152	0.189	0.752	−0.334	0.416
2	0.228	−0.152	0.076	0.5	−0.334	0.167
3	0.189	0	0.189	0.416	0	0.416
	0.758	−0.304	0.454	1.668	−0.668	1.0
	$B_1(Y^*, Y)$	$-B_1(R, Y)$	$I_1(Y)$	$\beta_1(Y^*, Y)$	$-\beta_1(R, Y)$	

TABLE 15. - *Contributions and relative contributions of the variates $X_j, R_j, X_{(j)}, Y^*$ and R to the synthetic index $I(Y)$. Example at hand*

j	$B\big(X_{(j)}, Y\big)$	$-B\big(R_j, Y\big)$	$B\big(X_j, Y\big)$	$\beta\big(X_{(j)}, Y\big)$	$-\beta\big(R_{(j)}, Y\big)$	$\beta\big(X_j, Y\big)$
1	0.402	−0.049	0.353	0.971	−0.118	0.853
2	0.193	−0.277	−0.084	0.466	−0.669	−0.203
3	0.206	−0.061	0.145	0.498	−0.147	0.350
	0.801	−0.387	0.414	1.935	−0.935	1.0
	$B(Y^*, Y)$	$-B(R, Y)$	$I(Y)$	$\beta(Y^*, Y)$	$-\beta(R, Y)$	

6. TWO TERMS ADDITIVE DECOMPOSITION OF $I_i(X_{(j)})$ AND $I(X_{(j)})$ AND THEIR RELATIONS WITH $B_i(X_j)$ AND $B(X_j)$

The point inequality index $I_i(X_{(j)})$ is now decomposed in two additive terms. From (15) and the relation $X_{(j)} = X_j + R_j$ we obtain:

$$I_i(X_{(j)}) = \frac{\overset{+}{M}_i(X_{(j)}) - \overset{-}{M}_i(X_{(j)})}{\overset{+}{M}_i(X_{(j)})} = B_i(X_j, X_{(j)}) + B_i(R_j, X_{(j)}), \tag{46}$$

where $B_i(X_j, X_{(j)}) = \frac{\overset{+}{M}_i(X_j) - \overset{-}{M}_i(X_j)}{\overset{+}{M}_i(X_{(j)})}$ and $B_i(R_j, X_{(j)}) = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(X_{(j)})}$, are the contributions of X_j and R_j to $I_i(X_{(j)})$. Averaging in (46) over i, gives:

$$\begin{aligned} I(X_{(j)}) &= B(X_j, X_{(j)}) + B(R_j, X_{(j)}), \text{ where} \\ B(X_j, X_{(j)}) &= \frac{1}{n} \cdot \sum_i B_i(X_j, X_{(j)}), \text{ and} \\ B(R_j, X_{(j)}) &= \frac{1}{n} \cdot \sum_i B_i(R_j, X_{(j)}). \end{aligned} \tag{47}$$

Table 16 illustrates the decompositions above proposed for the example at hand.

TABLE 16. - *Decomposition in two additive terms of $I_i(X_{(j)})$ and $I(X_{(j)})$*

i	$I_i(X_{(1)})$	$B_i(R_1, X_{(1)})$	$B_i(X_1, X_{(1)})$	$I_i(X_{(2)})$	$B_i(R_2, X_{(2)})$	$B_i(X_2, X_{(2)})$	$I_i(X_{(3)})$	$B_i(R_3, X_{(3)})$	$B_i(X_3, X_{(3)})$
1	0.789	0.352	0.438	0.789	0.528	0.261	0.555	0	0.575
2	0.735	0.196	0.539	0.625	0.704	−0.079	0.625	0	0.625
3	0.758	0	0.758	0.641	1.155	−0.514	0.555	0	0.555
4	0.750	0	0.750	0.583	1.082	−0.499	0.555	0.413	0.137
5	0.600	0	0.6	0.467	0.868	−0.401	0.444	0.333	0.111
	0.727	0.11	0.617	0.621	0.867	−0.245	0.547	0.15	0.397
	$I(X_{(1)})$	$B(R_1, X_{(1)})$	$B(X_1, X_{(1)})$	$I(X_{(2)})$	$B(R_2, X_{(2)})$	$B(X_2, X_{(2)})$	$I(X_{(3)})$	$B(R_3, X_{(3)})$	$B(X_3, X_{(3)})$

Now from the definitions of $B_i(X_j, X_{(j)})$, of $B_i(X_j, Y)$, and of $B_i(X_j, Y^*)$ we get:

$$\begin{cases} B_i(X_j, Y) &= B_i(X_j, Y^*) \cdot \delta_i, \\ B_i(X_j, Y^*) &= B_i(X_j, X_{(j)}) \cdot {}_j\overset{+}{\gamma}_i \\ B_i(X_j, Y) &= B_i(X_j, X_{(j)}) \cdot A_{ij} = \{I_i(X_{(j)}) - B_i(R_j, X_{(j)})\} \cdot A_{ij} \end{cases} \tag{48}$$

where the share ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$ is introduced in Section 4 , and where

$$\delta_i = \frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}, \text{ and } A_{ij} = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y)}. \tag{49}$$

In (48) are reported three relationships between the contributions of X_j to the point indexes $I_i(Y)$, $I_i(Y^*)$ and $I_i\left(X_{(j)}\right)$. Now, we ''extend'' these relations to the contributions of X_j to the synthetic indexes $I(Y)$, $I(Y^*)$ and $I\left(X_{(j)}\right)$. For this purpose we take the means of both sides of the first relation in (48):

$$\frac{1}{n}\cdot\sum_i B_i\left(X_j, Y\right) = \frac{1}{n}\cdot\sum_i B_i\left(X_j, Y^*\right)\cdot\delta_i.$$

Let ${}_j\tilde{\delta}$ be the Chisini (1929) mean of δ_i, that preserves the *l.h.s.* of the previous relation. Thus, $\frac{1}{n}\sum_i B_i\left(X_j, Y\right) = \frac{1}{n}\sum_i B_i\left(X_j, Y^*\right)\cdot{}_j\tilde{\delta} \Longrightarrow \quad {}_j\tilde{\delta} = \frac{1}{n}\sum_i B_i\left(X_j, Y\right)/\frac{1}{n}\sum_i B_i\left(X_j, Y^*\right)$

$$\begin{cases} {}_j\tilde{\delta} & = \frac{B\left(X_j,Y\right)}{B\left(X_j,Y^*\right)}, \quad \left(B\left(X_j, Y^*\right) > 0\right), \\ B\left(X_j, Y\right) & = B\left(X_j, Y^*\right)\cdot{}_j\tilde{\delta}. \end{cases} \tag{50}$$

Note that ${}_j\tilde{\delta}$ is the weighted mean of the ratios $\delta_i = \frac{\overset{+}{M}_i(Y^*)}{j\overset{+}{M}_i(Y)} = \frac{B_i\left(X_j,Y\right)}{B_i\left(X_j,Y^*\right)}$ with weights $B_i\left(X_j, Y^*\right)$. From the second relation in (48) we obtain another mean ${}_j\tilde{\gamma}$:

$$\frac{1}{n}\cdot\sum_i B_i\left(X_j, Y^*\right) = \frac{1}{n}\cdot\sum_i B_i\left(X_j, X_{(j)}\right)\cdot{}_j\overset{+}{\gamma}_i \Longrightarrow B\left(X_j, Y^*\right) =$$

$$\frac{1}{n}\cdot\sum_i B_i\left(X_j, X_{(j)}\right)\cdot{}_j\tilde{\gamma}.$$

Thus:

$$\begin{cases} {}_j\tilde{\gamma} & = \frac{B\left(X_j,Y^*\right)}{B\left(X_j,X_{(j)}\right)}, \\ B\left(X_j, Y^*\right) & = {}_j\tilde{\gamma}\cdot B\left(X_j, X_{(j)}\right). \end{cases} \tag{51}$$

Note that ${}_j\tilde{\gamma}$ is the weighted mean of the shares ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i\left(X_{(j)}\right)}{\overset{+}{M}_i(Y^*)} = \frac{B_i\left(X_j,Y^*\right)}{B_i\left(X_j,X_{(j)}\right)}$ with weights $B_i\left(X_j, X_{(j)}\right)$. Finally, from the third relation in (48) we have: $\frac{1}{n}\sum_i B_i\left(X_j, Y\right) = \frac{1}{n}\sum_i B_i\left(X_j, X_{(j)}\right)\cdot A_{ij} \Longrightarrow$

$$\begin{cases} {}_j\tilde{A} & = \frac{B\left(X_j,Y\right)}{B\left(X_j,X_{(j)}\right)}, \quad \left(B\left(X_j, X_{(j)}\right) > 0\right) \\ B\left(X_j, Y\right) & = {}_j\tilde{A}\cdot B\left(X_j, X_{(j)}\right). \end{cases} \tag{52}$$

So that ${}_j\tilde{A}$ is the Chisini mean of A_{ij} that preserves $B\left(X_j, Y\right)$; moreover ${}_j\tilde{A}$ ''coincides'' with the weighted mean of the ratios $A_{ij} = \frac{\overset{+}{M}_i\left(X_{(j)}\right)}{\overset{+}{M}_i(Y)} = \frac{B_i\left(X_j,Y\right)}{B_i\left(X_j,X_{(j)}\right)}$, with weights $B_i\left(X_j, X_{(j)}\right)$. Note that:

$${}_j\tilde{\gamma}\cdot{}_j\tilde{\delta} = {}_j\tilde{A}. \tag{53}$$

Now, substituting (53) in (52):

$$\begin{cases} B\left(X_j, X_{(j)}\right) \cdot {}_j\tilde{\gamma} \cdot {}_j\tilde{\delta} & = B\left(X_j, Y\right) \\ \left\{I\left(X_{(j)}\right) - B\left(R_j, X_{(j)}\right)\right\} \cdot {}_j\tilde{\gamma} \cdot {}_j\tilde{\delta} & = B\left(X_j, Y\right) \end{cases} \tag{54}$$

In conclusions, the relations (50), (51), (52), and (54) ''extend'' the relations between the contributions of X_j to the point inequality measures $I_i(Y)$, $I_i(Y^*)$ and $I_i\left(X_{(j)}\right)$, to the contributions of X_j to the synthetic inequality measure $I(Y)$, $I(Y^*)$ and $I\left(X_{(j)}\right)$.

The ratios A_{ij} and δ_i are reported in Table 17, while the shares ${}_j\overset{+}{\gamma}_i$ are reported in Table 7. Table 18 reports the computations to obtain the means ${}_j\tilde{\gamma}$, ${}_j\tilde{\delta}$, and ${}_j\tilde{A}$.

TABLE 17. - *Ratios* $A_{ij} = \dfrac{\overset{+}{M}_i\left(X_{(j)}\right)}{\overset{+}{M}_i(Y)}$

i	A_{i1}	A_{i2}	A_{i3}	$\sum_{j=1}^{3} A_{ij} = \delta_i = \dfrac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}$
1	0.432	0.288	0.341	1.061
2	0.464	0.291	0.364	1.118
3	0.537	0.317	0.366	1.22
4	0.682	0.341	0.409	1.432
5	0.682	0.341	0.409	1.432

TABLE 18. - *Contributions* $B\left(X_j, Y^*\right), B\left(X_j, X_{(j)}\right)$ *and* $B\left(X_j, Y\right)$, *and means* ${}_j\tilde{\gamma}$, ${}_j\tilde{\delta}$ *and* $\tilde{A}_j$

	$B\left(X_j, Y^*\right)$	$B\left(X_j, X_{(j)}\right)$	$B\left(X_j, Y\right)$	${}_j\tilde{\gamma}$	${}_j\tilde{\delta}$	$\tilde{A}_j$
j	(1)	(2)	(3)	$\frac{(1)}{(2)}$	$\frac{(3)}{(1)}$	$\frac{(3)}{(2)}$
1	0.276	0.617	0.353	0.447	1.278	0.572
2	−0.059	−0.246	−0.083	0.240	1.407	0.337
3	0.124	0.397	0.144	0.312	1.161	0.363
	0.341 $= B(Y, Y^*)$		0.414 $= I(Y)$			

Table 19 illustrates, for the example at hand, the finer decompositions of $I_i(Y)$ and $I(Y)$ obtained by the use of the relations

$$B_i\left(X_j, Y\right) = \left\{I_i\left(X_{(j)}\right) - B_i\left(R_j, X_{(j)}\right)\right\} \cdot A_{ij}$$

and

$$B(X_j, Y) = \{I(X_{(j)}) - B(R_j, X_{(j)})\} \cdot \tilde{A}_j.$$

TABLE 19. - *Finer representation of* $I_i(Y) = \sum_j \{I_i(X_{(j)}) - {}_jB_i(R_j, X_{(j)})\} \cdot A_{ij}$, *and of* $I(Y) = \sum_j \{I(X_{(j)}) - B(R_j, X_{((j))})\} \cdot \tilde{A}_j$

	$I_i(X_{(1)}) - B_i(R_1, X_{(1)}) = B_i(X_1, X_{(1)})$	A_{i1}	$B_i(X_1, Y)$	$I_i(X_{(2)}) - B_i(R_2, X_{(2)}) = B_i(X_2, X_{(2)})$	A_{i2}	$B_i(X_2, Y)$
i	(1)	(2)	(1) × (2)	(3)	(4)	(3) × (4)
1	0.790 − 0.351 = 0.439	0.432	0.189	0.789 − 0.527 = 0.262	0.288	0.076
2	0.735 − 0.196 = 0.539	0.464	0.250	0.626 − 0.704 = −0.078	0.291	−0.023
3	0.758 − 0.000 = 0.758	0.537	0.407	0.641 − 1.154 = −0.513	0.317	−0.163
4	0.750 − 0.000 = 0.750	0.682	0.512	0.583 − 1.083 = −0.500	0.341	−0.171
5	0.600 − 0.000 = 0.600	0.682	0.409	0.467 − 0.867 = −0.400	0.341	−0.136
	0.727 − 0.110 = 0.617 $I(X_{(1)}) - B(R_1, X_{(1)}) = B(X_1, X_{(1)})$	0.572 $\tilde{A}_1$	0.353 $B(X_1, Y)$	0.621 − 0.867 = −0.246 $I(X_{(2)}) - B(R_2, X_{(2)}) = B(X_2, X_{(2)})$	0.337 $\tilde{A}_2$	−0.083 $B(X_2, Y)$

	$I_i(X_{(3)}) - B_i(R_3, X_{(3)}) = B_i(X_3, X_{(3)})$	A_{i3}	$B_i(X_3, Y)$	$I_i(Y) = \sum_j B_i(X_j, Y)$
i	(5)	(6)	(5) × (6)	(1) × (2) + (3) × (4) + (5) × (6)
1	0.555 − 0.000 = 0.555	0.341	0.189	0.455
2	0.625 − 0.000 = 0.625	0.364	0.227	0.455
3	0.555 − 0.000 = 0.555	0.366	0.203	0.447
4	0.555 − 0.418 = 0.138	0.409	0.057	0.398
5	0.444 − 0.333 = 0.111	0.409	0.045	0.318
	0.547 − 0.150 = 0.397 $I(X_{(3)}) - B(R_3, X_{(3)}) = B(X_3, X_{(3)})$	0.363 $\tilde{A}_3$	0.144 $B(X_3, Y)$	0.414 $I(Y)$

Interesting informations are reported in Table 19. For example, to obtain the contribution $B(X_1, Y) = 0.353$ we have, first of all, to subtract $B(R_1, X_{(1)}) = 0.110$ to the synthetic inequality index $I(X_{(1)}) = 0.727$. The value obtained is the contribution of X_1 to $I(X_{(1)}) : B(X_1, X_{(1)}) = 0.617$. Then, to obtain the contribution $B(X_1, Y)$ we multiply this last value by the mean value $\tilde{A}_1 = 0.572$. More details are obtained substituting $\tilde{A}_1$ with ${}_1\tilde{\gamma} \cdot {}_1\tilde{\delta}$. In conclusion:

$$\begin{aligned} \{I(X_{(1)}) - B(R_1, X_{(1)})\} \cdot {}_1\tilde{\gamma} \cdot {}_1\tilde{\delta} &= B(X_1, Y) \\ \{0.727 - 0.110\} \cdot 0.447 \cdot 1.278 &= 0.353 \end{aligned}$$

In other words the contribution $B(X_1, Y)$ is function of: the inequality index

$I(X_{(1)})$, the contribution of R_1 to $I(X_{(1)})$, the mean ${}_1\tilde{\gamma}$ of the shares ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$, and of the mean ${}_1\tilde{\delta}$ of the ratios $\delta_i = \frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}$. Note that the ratios δ_i are related to the association among the sources.

7. COMPARISON OF THE DECOMPOSITION BY SOURCES OF THE GINI AND ZENGA INEQUALITY INDEXES

In the previous sections we have illustrated, first of all, the following additive decomposition by sources of the Zenga point index: $I_i(Y) = \sum_{j=1}^{c} B_i(X_j, Y)$. Then, in a natural way, we have derived the decomposition of the synthetic index: $I(Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(Y) = \sum_{j=1}^{c} \frac{1}{n} \cdot \sum_{i=1}^{n} B_i(X_j, Y) = \sum_{j=1}^{c} B(X_j, Y)$ Viceversa, for the synthetic Gini index $G(Y)$ many decompositions by sources have been obtained by different mathematical expressions of $G(Y)$. Now we will analyse three of these decompositions and compare them with the decomposition of $I(Y)$. In this section we assume that the c variates X_j are non-negative and that their mean values are positive.

Gini's mean difference of the variate X that takes the values $x_1, \ldots, x_i, \ldots, x_n$ on the n units of a finite population is given by: $\Delta(X) = \frac{1}{n(n-1)} \sum_i \sum_j |x_i - x_j|$. It is well known (Gini, 1914) that $S(X) = \sum_i \sum_j |x_i - x_j| = 2 \sum_i x_{(i)}(2i - n - 1)$. The Gini concentration ratio of X can be evaluated by $G(X) = \frac{\Delta(X)}{2 \cdot M(X)}$, where $M(X)$ is the mean of X. In the case of the sum $Y = \sum_j X_j$, we have:

$$S(Y) = 2 \sum_i y_{(i)} \cdot (2i - n - 1), \tag{55}$$

and substituting (2) in (55) yields:

$$S(Y) = 2 \sum_i \left(\sum_j x_{ij} \right) \cdot (2i - n - 1) = \sum_j 2 \sum_i x_{ij}(2i - n - 1). \tag{56}$$

Thus, the Gini concentration ratio of the sum Y can be written as

$$G(Y) = \frac{\Delta(Y)}{2 \cdot M} = \frac{2 \sum_i y_{(i)}(2i - n - 1)}{n \cdot (n-1) \cdot 2M} = \sum_j \frac{\sum_i x_{ij}(2i - n - 1)}{n(n-1)M} =$$

$$= \sum_j C(X_j), \tag{57}$$

where $C(X_j) = \frac{\sum_i x_{ij}(2i-n-1)}{n(n-1)M}$, is the contribution of X_j to $G(Y)$. The relative contributions are given by $\lambda(X_j) = \frac{C(X_j)}{G(Y)}$.

Table 20 calculates, for the example at hand, $G(Y)$ and the contributions $C(X_j)$ and $\lambda(X_j)$.

TABLE 20. - *Gini index $G(Y)$ and contributions $C(X_j)$ and $\lambda(X_j)$*

i	x_{i1}	x_{i2}	x_{i3}	$y_{(i)}$	$(2i-6)$	$x_{i1}(2i-6)$	$x_{i2}(2i-6)$	$x_{i3}(2i-6)$	$y_{(i)}(2i-6)$
1	7	6	5	18	−4	−28	−24	−20	−72
2	6	11	5	22	−2	−12	−22	−10	−44
3	3	15	10	28	0	0	0	0	0
4	14	6	18	38	+2	28	12	36	76
5	30	2	12	44	+4	120	8	48	176
Tot.	60	40	50	150	0	108	−26	54	136
						$C(X_1)$	$C(X_2)$	$C(X_3)$	$G(Y)$
						0.18	−0.043	0.09	0.227
						$\lambda(X_1)$	$\lambda(X_2)$	$\lambda(X_3)$	
						0.794	−0.191	0.397	1.000

We are ready to compare the relative contributions to $G(Y)$ and $I(Y)$ of the variates X_1, X_2 and X_3.

Table 21 shows that the differences between the relative contributions of the two indexes is not negligible for X_3 and X_1, however their values are ‘‘coherent’’ w.r.t. the correspondent shares. As a matter of fact, for both indexes the variates X_1 and X_3 increase the inequality while X_2 decreases it. For more details on this point see Zenga (2013) Section 6.

TABLE 21. - *Shares and relative contributions of X_1, X_2 and X_3 to G and I*

j	1	2	3
$\lambda(X_j)$	0.794	−0.191	0.397
$\beta(X_j)$	0.856	−0.201	0.349
${}_j\gamma$	0.4	0.267	0.333

To have an idea of the values that $C(X_j)$ can assume, we first introduce the following Definition. The values $x_1, ..., x_i, ..., x_n$ of X, are arranged in non-increasing order if $x_i = x_{(n-i+1)}$, $(i = 1, 2, ..., n)$.

Now by theorem 368 of Hardy, Littlewood, Polya (1952, p. 261) we have

$$\sum_i i \cdot x_{(n-i+1)} \leq \sum_i i \cdot x_i \leq \sum_i i \cdot x_{(i)}. \tag{58}$$

The value in the l.h.s. of (58) is provided by:

$$\sum_{t=n}^{1}(n+1-t)\cdot x_{(t)}=(n+1)\cdot n\cdot M(X)-\sum_{i=1}^{n} i\cdot x_{(i)}. \tag{59}$$

Utilizing (59) in (58) gives:

$$2(n+1)\cdot n\cdot M(X)-\sum_i 2\cdot i\cdot x_{(i)}\le\sum_i 2\cdot i\cdot x_i\le\sum_i 2\cdot i\cdot x_{(i)}.$$

Subtracting $(n+1)\cdot n\cdot M(X)$ from each term of the above relation yields $-\sum_i x_{(i)}(2i-n-1)\le\sum_i x_i(2i-n-1)\le\sum_i x_{(i)}(2i-n-1)\Rightarrow$

$$-\frac{S(X)}{2}\cdot\frac{1}{n\cdot(n-1)}\cdot\frac{M(X)}{M(X)}\le\frac{\sum x_i(2i-n-1)}{n(n-1)\cdot}\le\frac{S(X)}{2}\cdot\frac{1}{n\cdot(n-1)}\cdot\frac{M(X)}{M(X)}\Rightarrow$$

$$-G(X)\cdot M(X)\le\frac{\sum_i x_i(2i-n-1)}{n(n-1)}\le G(X)\cdot M(X). \tag{60}$$

Therefore, for the component X_j of Y we have

$$-G(X_j)\cdot\frac{{}_jM}{M}\le\frac{\sum_i x_{ij}(2i-n-1)}{n(n-1)\cdot M}\le G(X_j)\cdot\frac{{}_jM}{M}\Longrightarrow$$

$$-G(X_j)\cdot{}_j\gamma\le C(X_j)\le G(X_j)\cdot{}_j\gamma, \tag{61}$$

where $G(X_j)=\frac{\Delta(X_j)}{2\cdot{}_jM}$ is the Gini index of the component X_j of Y. Relation (61) informs that if in the data matrix (Table 1) the values x_{ij} are arranged in:

- non-decreasing order (X_j is cograduated with Y) $\Rightarrow C(X_j)=G(X_j)\cdot{}_j\gamma$;
- non-increasing order (X_j is contra-graduated with Y) $\Rightarrow C(X_j)=-G(Xj)\cdot{}_j\gamma$.

The reordering variates, widely used in the previous sections to decompose Zenga's index, have a central role in some decompositions by sources of Gini index, too. To this purpose we will now illustrate the decompositions of $G(Y)$ proposed by Radaelli, Zenga (2002, 2005), by Rao (1969), and by Lerman, Yitzhaki (1984, 1985).

From (56) and the identity $x_{ij}=x_{(ij)}-r_{ij}$,$(i=1,\dots,n)$, we obtain

$$\begin{aligned}S(Y)&=\sum_j 2\sum_i x_{(ij)}\cdot(2i-n-1)-\sum_j 2\sum_i r_{ij}(2i-n-1)\\&=\sum_j S(X_j)-4\sum_j\sum_i r_{ij}\cdot i\end{aligned} \tag{62}$$

From (58) it derives that $\sum_i i(x_{(ij)}-x_{ij})=\sum_i i\cdot r_{ij}\ge 0,(j=1,\dots,c)$, with equality only if $x_{ij}=x_{(ij)},(i=1,\dots,n)$.

Dividing (62) first by $n\cdot(n-1)$ and then by $2M$ gives the following decomposition for $\Delta(Y)$ and $G(Y)$, Radaelli *et al.* (2005):

$$\Delta(Y)=\sum_j\Delta(X_j)-\frac{4}{n(n-1)}\sum_j\sum_i i\cdot r_{ij}\Rightarrow$$

$$\frac{\Delta(Y)}{2M} = \sum_j \frac{\Delta(X_j)}{2 \cdot {}_jM} \cdot \frac{{}_jM}{M} - \sum_j \sum_i \frac{4 \cdot i \cdot r_{ij}}{n \cdot (n-1) \cdot 2M} \Longrightarrow$$

$$G(Y) = \sum_j G(X_j) \cdot_j \gamma - \frac{2}{(n-1) \cdot M} \cdot \sum_j \frac{1}{n} \sum_i i \cdot r_{ij}. \tag{63}$$

Equation (63) is an easy decomposition that allows Gini's inequality index of a sum to be provided by the difference of the weighted arithmetic mean of the Gini indexes of each component with a non negative term that is the contribution to $G(Y)$ of all the reordering variates $R_j, (j = 1, ..., c)$. Finally, from (57) and (63) we have the following representation:

$$C(X_j) = \left\{ G(X_j) - \frac{2}{(n-1) \cdot {}_jM} \cdot \frac{1}{n} \sum_i i \cdot r_{ij} \right\} \cdot {}_j\gamma. \tag{64}$$

Formula (64) is similar to the representations (48) of $B_i(X_j, Y)$ and (54) of $B(X_j, Y)$.

To illustrate the Rao (1969) decomposition of $G(Y)$ we point out that $G(Y)$ and $G(X_j)$ can be obtained by the following relations (Gini, 1914):

$$G(Y) = \frac{1}{\sum_{i=1}^{n-1} \frac{i}{n}} \cdot \sum_{i=1}^{n-1} \left(\frac{i}{n} - \frac{Q_i(Y)}{n \cdot M} \right), \tag{65}$$

$$G(X_j) = \frac{1}{\sum_{i=1}^{n-1} \frac{i}{n}} \cdot \sum_{i=1}^{n-1} \left(\frac{i}{n} - \frac{Q_i(X_{(j)})}{n \cdot_j M} \right). \tag{66}$$

Expression (66) is also applied by Rao on the cumulative sum $Q_i(X_j)$, obtaining the statistics

$$G'(X_j) = \frac{1}{\sum_{i=1}^{n-1} \frac{i}{n}} \cdot \sum_{i=1}^{n-1} \left(\frac{i}{n} - \frac{Q_i(X_j)}{n \cdot_j M} \right). \tag{67}$$

Note that $G'(X_j)$ is not Gini's concentration ratio since the n values x_{ij} are not necessarily arranged in non-decreasing order. Rao shows that:

$$-G(X_j) \le G'(X_j) \le G(X_j),$$

and that

$$G(Y) = \sum_j G(X_j) \cdot_j \gamma - \sum_j G(X_j) \cdot_j \gamma \cdot \left(1 - \frac{G'(X_j)}{G(X_j)} \right). \tag{68}$$

Comparing decompositions (68) and (63) we observe that for both we have to subtract a non negative term from the weighted arithmetic mean of the concentration ratios computed on each component. These terms are

$$\sum_j G(X_j) \cdot_j \gamma \cdot \left(1 - \frac{G'(X_j)}{G(X_j)}\right) = \sum_j {}_j\gamma \cdot \left(G(X_j) - G'(X_j)\right), \tag{69}$$

and

$$\frac{2}{(n-1) \cdot M} \cdot \sum_j \frac{1}{n} \sum_i i \cdot r_{ij}. \tag{70}$$

Obviously (69) is equal to (70). It is possible, starting from (69), to obtain directly this equality.

To illustrate the Lerman *et al.* (1984, 1985) decomposition of $G(Y)$ we introduce: the variate P that assumes the values $(1, 2, ..., n)$; the variates $\{(X_{(j)}, P), (X_j, P), (R_j, P)\}, \forall(j)$, that assume the values $\{(x_{(ij)}, i), (x_{ij}, i), (r_{ij}, i);\ i = 1, 2, ..., n\}$; and the variates $\{(Y^*, P), (Y, P), (R, P)\}$ that assume the values $\left\{\left(y^*_{(i)}, i\right), (y_{(i)}, i), (r_i, i);\ i = 1, 2, ..., n\right\}$. Now, it is easy to show that:

$$Cov(X_j, P) = \frac{1}{n} \sum_i (x_{ij} -_j M)\left(i - \frac{n+1}{2}\right) = \frac{1}{2n} \sum_i x_{ij}(2i - n - 1); \tag{71}$$

$$Cov(X_{(j)}, P) = \frac{1}{2n} \sum_i x_{(ij)}(2i - n - 1) = \frac{n-1}{4} \Delta(X_j); \tag{72}$$

$$Cov(R_j, P) = \frac{1}{2n} \sum_i r_{ij}(2i - n - 1) = \frac{1}{n} \sum_i r_{ij} \cdot i; \tag{73}$$

$$Cov(Y, P) = \frac{1}{2n} \sum_i y_{(i)}(2i - n - 1) = \frac{n-1}{4} \cdot \Delta(Y); \tag{74}$$

$$Cov(Y^*, P) = \frac{1}{2n} \sum_i y^*_{(i)}(2i - n - 1) = \frac{n-1}{4} \cdot \Delta(Y^*); \tag{75}$$

$$Cov(R, P) = \frac{1}{2n} \sum_i r_i(2i - n - 1) = \frac{1}{n} \sum_i r_i \cdot i. \tag{76}$$

The relationship between the covariance and the Gini mean difference reported above, was obtained by De Vergottini (1950).

Now we illustrate the decomposition of $G(Y)$ proposed by Lerman *et al.* (1984, 1985).

$$\begin{aligned} G(Y) &= \frac{\Delta(Y)}{2M} = \frac{4 \cdot Cov(Y, P)}{(n-1) \cdot 2M} = \frac{4}{(n-1) \cdot 2M} \cdot Cov[(X_1 + ... + X_c), P] \\ &= \sum_j \frac{4}{(n-1) \cdot 2M} Cov(X_j, P) = \sum_j \frac{4}{(n-1) \cdot 2M} \frac{Cov(X_j, P)}{Cov(X_{(j)}, P)} \cdot Cov(X_{(j)}, P) \\ &= \sum_j \frac{\Delta(X_j)}{2 \cdot_j M} \cdot \Gamma_j \cdot_j \gamma = \sum_j G(X_j) \cdot \Gamma_j \cdot_j \gamma = \sum_j C(X_j), \end{aligned} \tag{77}$$

where the contribution $C(X_j)$ is equal to the product of $G(X_j)$, the share ${}_j\gamma$ and the ''so called'' Gini correlation coefficient

$$\Gamma_j = \frac{Cov(X_j, P)}{Cov(X_{(j)}, P)} = \frac{\sum_i x_{ij}(2i - n - 1)}{\sum_i x_{(ij)}(2i - n - 1)} \tag{78}$$

between the source and the rank of total Y.

As it was pointed out in Radaelli *et al.* (2005), Γ_j coincides with Rao's ratio $\frac{G'(X_j)}{G(X_j)}$. Thus, substituting in (79) Γ_j with $\frac{G'(X_j)}{G(X_j)}$ we obtain the (Rao, 1969) decomposition (68):

$$\begin{cases} G(Y) = \sum_j G(X_j) \cdot \frac{G'(X_j)}{G(X_j)} \cdot {}_j\gamma = \sum_j \Big(G(X_j) - \big(G(X_j) - G'(X_j)\big)\Big) \cdot {}_j\gamma \\ \qquad = \sum_j G(X_j) \cdot {}_j\gamma - \sum_j {}_j\gamma \cdot \big(G(X_j) - G'(X_j)\big). \end{cases}$$

The decomposition of $G(Y)$ proposed by Rao (1969), by Lerman *et al.* (1984, 1985) and by Radaelli *et al.* (2005) are obviously equivalent but they are different in the way they have been obtained and in their interpretation. The interpretation is clear in (63) where the sum of the individual weighted differences (reordering values) $\sum_{i=1}^{n} \big(x_{(ij)} - x_{ij}\big) \cdot i = \sum_{i=1}^{n} r_{ij} \cdot i$ is used. (Note that: $\sum_j \frac{1}{n} \sum_i r_{ij} \cdot i =$ $= \sum_j Cov(R_j, P) = Cov(R, P)$).

In the Rao decomposition the interpretation is not clear because we have to compute ''concentration ratios'' $G'(X_j)$ on values in non increasing order and obtain values which may be negative for a measure that by definition should lie in the interval $[0; 1]$. In the case of the Lerman *et al.* (1984, 1985) decomposition, the use of the ''so called'' Gini correlation coefficient between the source and the rank of the total Y introduces some confusion in the literature. In fact, Γ_j is the ratio between $Cov(X_j, P)$ and $Cov(X_{(j)}, P)$ while the correlation coefficient between $X_{(j)}$ and P is given by $r(X_{(j)}, P) = \frac{Cov(X_{(j)}, P)}{\sigma(X_j) \cdot \sigma(P)}$, where $\sigma(X_j)$ and $\sigma(P)$ are the standard deviation of X_j and P respectively. The correlation coefficient $r(X_{(j)}, P)$ was introduced by De Vergottini (1950) who employed it in the following popular expression:

$$G(X_j) = \sqrt{\frac{n+1}{n}} \cdot \sqrt{\frac{1}{3}} \cdot \frac{\sigma(X_j)}{{}_jM} \cdot r(X_{(j)}, P).$$

For this point see: Stuart (1954), Piesche (1975), Dancelli (1987), Zenga (1987).

There is a vast literature on the decomposition of the Gini inequality index. In this section we have analized only the decomposition proposed by Rao (1969), by Lerman *et al.* (1984, 1985) and by Radaelli *et al*. (2005), because they are similar to the decomposition by sources of Zenga index proposed in this paper. More details on the decompositions of Gini index can be found in Silber (1989), in Radaelli (2010), in Giorgi (2011), and in Zenga (2013).

8. APPLICATION

The approach proposed in Zenga *et al.* (2012) for the decompositions $I_i(Y) = \sum_{j=1}^{c} B_i(X_j, Y)$ and $I(Y) = \sum_{j=1}^{c} B(X_j, Y)$, is extended in Zenga (2013) for the decompositions of the Gini (1914) and Bonferroni (1930) indexes, too. In this latter paper it is shown (Lemma 1) that the relative contributions of X_j to the inequality point measure of the three indexes are equal. In both papers the proposed decompositions are applied to the data-set supplied by the 2008 Central Bank of Italy samples survey on household income and wealth (Bank of Italy, 2010). These applications confirme that the relative contributions of the income sources X_j to the point and synthetic values of the Gini, Bonferroni and Zenga indexes are ''coherent'' with the results of the methodology proposed in the above mentioned two articles.

In this section we apply the decompositions related to the reordering variates $R_j = X_{(j)} - X_j$ to the same data used in Zenga *et al.* (2012). This data set covers $n = 7642$ households with $x_{ij} \geq 0, \forall(i,j)$. For each household we have the following variates: X_1 (Payroll income), X_2 (Pensions and net transfers), X_3 (Net self-employment income), X_4 (Property income), and $Y = \sum_{j=1}^{4} X_j$ (Net disposable income).

We remark that the decompositions evaluated in this section are based on the following lower means $\left(\overset{-}{M}_i(X_j), \overset{-}{M}_i(X_{(j)})\right)$ and upper means $\left(\overset{+}{M}_i(X_j), \overset{+}{M}_i(X_{(j)})\right)$, $(j = 1, 2, 3, 4; i = 1, \ldots, 7642)$. To have an idea of the values of these means Table 22 reports the upper and the lower means of the variates X_j and $X_{(j)}$, and of the totals Y and Y^*, for the following percentiles

$$p : (p = 0.05; 0.10; 0.25; 0.50; 0.75; 0.90; 0.95).$$

Moreover, the last rows of Table 22 report the shares ${}_j\overset{+}{\gamma}_{(p)}$ and the ratios $\delta_{(p)} = \overset{+}{M}_{(p)}(Y^*)/\overset{+}{M}_{(p)}(Y)$.

TABLE 22. - *Upper and lower means of the variates X_j and $X_{(j)}$, and of the totals Y and Y^*; shares ${}_j\overset{+}{\gamma}_{(p)}$ and ratios $\delta_{(p)}$*

	percentiles p 0.05	0.10	0.25	0.50	0.75	0.90	0.95
$\overset{+}{M}_{(p)}(X_1)$	12360.4	12930.9	14709.5	18437.6	23788.3	28535.3	29976.3
$\overset{-}{M}_{(p)}(X_1)$	1027.1	1558.7	3046.4	5149.9	7795.5	9933.5	10836.7

(*follows*)

$\overset{+}{M}_{(p)}(X_{(1)})$	12414.4	13104.1	15725.0	23583.1	33491.4	45284.7	54229.0
$\overset{-}{M}_{(p)}(X_{(1)})$	0.0	0.0	0.0	4.4	4561.2	8072.5	9560.3
$\overset{+}{M}_{(p)}(X_2)$	10389.1	10605.3	11164.6	12274.4	14118.6	17944.0	23642.4
$\overset{-}{M}_{(p)}(X_2)$	4180.4	5338.3	6820.8	7882.9	8731.9	9204.6	9364.7
$\overset{+}{M}_{(p)}(X_{(2)})$	10609.1	11198.5	13438.2	18958.1	25741.2	35370.0	44443.2
$\overset{-}{M}_{(p)}(X_{(2)})$	0.0	0.0	0.0	1199.2	4857.8	7268.5	8270.0
$\overset{+}{M}_{(p)}(X_3)$	3791.3	3978.9	4673.3	6381.3	10169.9	16963.3	23988.4
$\overset{-}{M}_{(p)}(X_3)$	214.5	314.4	429.8	843.6	1426.6	2129.0	2540.0
$\overset{+}{M}_{(p)}(X_{(3)})$	3802.6	4013.8	4816.6	7224.9	14449.8	31773.4	44321.5
$\overset{-}{M}_{(p)}(X_{(3)})$	0.0	0.0	0.0	0.0	0.0	483.4	1470.0
$\overset{+}{M}_{(p)}(X_4)$	7434.6	7740.0	8699.7	10610.5	14023.4	19898.8	24969.4
$\overset{-}{M}_{(p)}(X_4)$	1348.5	1642.4	2422.1	3650.0	4832.5	5711.5	6191.4
$\overset{+}{M}_{(p)}(X_{(4)})$	7505.5	7921.3	9318.0	11825.4	16529.3	24717.9	33052.0
$\overset{-}{M}_{(p)}(X_{(4)})$	0.0	10.9	567.2	2435.1	3997.2	5176.1	5765.0
$\overset{+}{M}_{(p)}(Y)$	33975.3	35255.2	39247.0	47703.8	62100.3	83342.4	102576.5
$\overset{-}{M}_{(p)}(Y)$	6770.4	8853.9	12719.1	17526.3	22786.6	26978.7	28932.9
$\overset{+}{M}_{(p)}(Y^*)$	34331.6	36237.7	43297.7	61591.4	90211.7	137146.0	176045.7
$\overset{-}{M}_{(p)}(Y^*)$	0.0	10.9	567.2	3638.6	13416.2	21000.5	25066.1
${}_1\overset{+}{\gamma}_{(p)}$	0.362	0.362	0.368	0.383	0.371	0.330	0.308
${}_2\overset{+}{\gamma}_{(p)}$	0.309	0.309	0.310	0.308	0.285	0.258	0.252
${}_3\overset{+}{\gamma}_{(p)}$	0.111	0.111	0.111	0.117	0.160	0.232	0.252
${}_4\overset{+}{\gamma}_{(p)}$	0.219	0.219	0.215	0.192	0.183	0.180	0.188
$\delta_{(p)}$	1.01	1.028	1.103	1.291	1.453	1.645	1.716

8.1 *Bivariate decompositions*

In this section we analyze the bivariate decompositions (Section 5.1) of $I_{(p)}(Y^*)$, $I(Y^*)$ and of $I_{(p)}(Y)$ and $I(Y)$.

Table 23 reports the contributions of $X_{(j)}$, X_j and R_j to the bivariate decomposition of $I_{(p)}(Y^*)$, and $I(Y^*)$. Table 24 reports the relative contributions of the reordering variates to $I_{(p)}(Y^*)$ and $I(Y^*)$.

TABLE 23. - *Contributions of the variates $X_j, X_{(j)}$ and R_j, $(j = 1, 2, 3, 4)$, to the two additive terms representation of $I(Y^*) = [B(Y, Y^*) + B(R, Y^*)]$, and $I_{(p)}(Y^*) = \left[B_{(p)}(Y, Y^*) + B_{(p)}(R, Y^*)\right]$*

	percentiles p								
	0.05	0.10	0.25	0.50	0.75	0.90	0.95		
$B_{(p)}(X_1, Y^*)$	0.330	0.314	0.269	0.216	0.176	0.136	0.109	0.213	$B(X_1, Y^*)$
$B_{(p)}(R_1, Y^*)$	0.032	0.048	0.094	0.167	0.145	0.135	0.145	0.128	$B(R_1, Y^*)$
$B_{(p)}(X_{(1)}, Y^*)$	0.362	0.362	0.363	0.383	0.321	0.271	0.254	0.341	$B(X_{(1)}, Y^*)$
$B_{(p)}(X_2, Y^*)$	0.181	0.145	0.101	0.071	0.059	0.064	0.081	0.094	$B(X_2, Y^*)$
$B_{(p)}(R_2, Y^*)$	0.128	0.164	0.209	0.217	0.172	0.141	0.124	0.179	$B(R_2, Y^*)$
$B_{(p)}(X_{(2)}, Y^*)$	0.309	0.309	0.310	0.288	0.231	0.205	0.205	0.273	$B(X_{(2)}, Y^*)$
$B_{(p)}(X_3, Y^*)$	0.104	0.101	0.098	0.090	0.096	0.108	0.122	0.096	$B(X_3, Y^*)$
$B_{(p)}(R_3, Y^*)$	0.007	0.010	0.013	0.027	0.064	0.120	0.121	0.047	$B(R_3, Y^*)$
$B_{(p)}(X_{(3)}, Y^*)$	0.111	0.111	0.111	0.117	0.160	0.228	0.243	0.143	$B(X_{(3)}, Y^*)$
$B_{(p)}(X_4, Y^*)$	0.177	0.168	0.145	0.113	0.101	0.103	0.107	0.121	$B(X_4, Y^*)$
$B_{(p)}(R_4, Y^*)$	0.042	0.050	0.057	0.039	0.038	0.039	0.048	0, 049	$B(R_4, Y^*)$
$B_{(p)}(X_{(4)}, Y^*)$	0.219	0.218	0.202	0.152	0.139	0.142	0.155	0.170	$B(X_{(4)}, Y^*)$
$B_{(p)}(Y, Y^*)$	0.793	0.728	0.613	0.490	0.433	0.411	0.419	0.524	$B(Y, Y^*)$
$B_{(p)}(R, Y^*)$	0.207	0.272	0.374	0.450	0.418	0.436	0.439	0.403	$B(R, Y^*)$
$I_{(p)}(Y^*)$	1.00	1.00	0.987	0.940	0.851	0.847	0.858	0.927	$I(Y^*)$

The last row of Table 23 shows that for $p \leq 0.25$, $I_{(p)}(Y^*) \cong 1$, and that for $p > 0.25$, $I_{(p)}(Y^*)$ decreases slowly.

TABLE 24. - *Relative contributions of the reordering variates to the indexes $I_{(p)}(Y^*)$ and $I(Y^*)$*

	percentiles p								
	0.05	0.10	0.25	0.50	0.75	0.90	0.95		
$\beta_{(p)}(R_1, Y^*)$	0.032	0.048	0.095	0.177	0.170	0.159	0.169	0.138	$\beta(R_1, Y^*)$
$\beta_{(p)}(R_2, Y^*)$	0.128	0.164	0.211	0.231	0.202	0.166	0.145	0.193	$\beta(R_2, Y^*)$
$\beta_{(p)}(R_3, Y^*)$	0.007	0.01	0.013	0.028	0.075	0.142	0.140.	0.050	$\beta(R_3, Y^*)$
$\beta_{(p)}(R_4, Y^*)$	0.042	0.05	0.058	0.041	0.045	0.046	0.056	0.053	$\beta(R_4, Y^*)$
$\beta_{(p)}(R, Y^*)$	0.207	0.272	0.378	0.479	0.491	0.515	0.512	0.435	$\beta(R, Y^*)$

The relative contributions of R_j and R to $I_{(p)}(Y^*)$, reported in Table 24, can be represented by:

$$\beta_{(p)}(R_j, Y^*) = \left[B_{(p)}(R_j, Y^*)/I_{(p)}(Y^*)\right] = \frac{\overset{+}{M}_{(p)}(R_j) - \overset{-}{M}_{(p)}(R_j)}{\overset{+}{M}_{(p)}(Y^*) - \overset{-}{M}_{(p)}(Y^*)}, \tag{79}$$

and

$$\beta_{(p)}(R, Y^*) = \left[B_{(p)}(R, Y^*)/I_{(p)}(Y^*)\right] = \frac{\overset{+}{M}_{(p)}(R) - \overset{-}{M}_{(p)}(R)}{\overset{+}{M}_{(p)}(Y^*) - \overset{-}{M}_{(p)}(Y^*)} \ . \tag{80}$$

Moreover, we may represent the relative contribution of R_j and R to the synthetic index $I(Y^*)$ by:

$$\beta(R_j, Y^*) = \frac{B(R_j, Y^*)}{I(Y^*)} = \frac{\frac{1}{n} \cdot \sum_i B_i(R_j, Y^*)}{I(Y^*)} = \sum_{i=1}^{n} \beta_i(R_j, Y^*) \cdot \frac{I_i(Y^*)}{n \cdot I(Y^*)}, \tag{81}$$

$$\beta(R, Y^*) = \frac{B(R, Y^*)}{I(Y^*)} = \frac{\frac{1}{n} \cdot \sum_i B_i(R, Y^*)}{I(Y^*)} = \sum_{i=1}^{n} \beta_i(R, Y^*) \cdot \frac{I_i(Y^*)}{n \cdot I(Y^*)}. \tag{82}$$

Table 24 informs that $\beta_{(0.75)}(R, Y^*) = 0.491$. According to formula (80) this means that the difference $\left[\overset{+}{M}_{(0.75)}(R) - \overset{-}{M}_{(0.75)}(R)\right]$ is the 49.1% of the difference $\left[\overset{+}{M}_{(0.75)}(Y^*) - \overset{-}{M}_{(0.75)}(Y^*)\right]$, and that the difference $\left[\overset{+}{M}_{(0.75)}(Y) - \overset{-}{M}_{(0.75)}(Y)\right]$ is the 50.9% of the remaining part of $\left[\overset{+}{M}_{(0.75)}(Y^*) - \overset{-}{M}_{(0.75)}(Y^*)\right]$.The last row of Table 24 shows that the relative contribution $\beta_{(p)}(R, Y^*)$, of the (total) reordering variate R to the point inequality $I_{(p)}(Y^*)$, is an empirical quasi-increasing function of the percentile p. The relative contribution of R to the synthetic index $I(Y^*)$ is the 43.5% of $I(Y^*) = 0.927$. In other words, there is a great ''compensation'' among the components of the household total income Y. The greatest portion of this compensation is due to the reordering variate R_2 (Pensions and net transfers). Viceversa, in this compensation context, the role of the reordering variate R_4 (Property income) is negligible and quasi-constant for all the percentiles p.

Now we will analyse, (Tables 25 and 26), the role of the reordering variates in the decomposition of the inequality of the total income Y.

TABLE 25. - *Contributions of the variates $X_j, X_{(j)}$ and $R_j, (j = 1,2,3,4)$, to the two subtractive terms representation of $I(Y) = [B(Y^*, Y) - B(R, Y)]$, and $I_{(p)}(Y) = \left[B_{(p)}(Y^*, Y) - B_{(p)}(R, Y)\right]$*

	percentiles p 0.05	0.10	0.25	0.50	0.75	0.90	0.95		
$B_{(p)}\left(X_{(1)}, Y\right)$	0.366	0.372	0.400	0.494	0.496	0.446	0.436	0.440	$B\left(X_{(1)}, Y\right)$
$-B_{(p)}(R_1, Y)$	−0.032	−0.049	−0.103	−0.215	−0.211	−0.223	−0.249	−0.174	$-B(R_1, Y)$
$B_{(p)}(X_1, Y)$	0.334	0.323	0.297	0.279	0.258	0.223	0.187	0.266	$B(X_1, Y)$
$B_{(p)}\left(X_{(2)}\right)$	0.312	0.318	0.342	0.372	0.338	0.337	0.352	0.361	$B\left(X_{(2)}, Y\right)$
$-B_{(p)}(R_2, Y)$	−0.129	−0.169	−0.231	−0.280	−0.251	−0.232	−0.213	−0.232	$-B(R_2, Y)$
$B_{(p)}(X_2, Y)$	0.183	0.149	0.111	0.092	0.087	0.105	0.139	0.128	$B(X_2, Y)$
$B_{(p)}\left(X_{(3)}, Y\right)$	0.112	0.114	0.122	0.151	0.234	0.375	0.417	0.210	$B\left(X_{(3)}, Y\right)$
$-B_{(p)}(R_3, Y)$	−0.007	−0.01	−0.014	−0.035	−0.093	−0.197	−0.208	−0.083	$-B(R_3, Y)$
$B_{(p)}(X_3, Y)$	0.105	0.104	0.108	0.116	0.141	0.178	0.209	0.127	$B(X_3, Y)$
$B_{(p)}\left(X_{(4)}, Y\right)$	0.221	0.224	0.223	0.196	0.203	0.234	0.266	0.220	$B\left(X_{(4)} Y\right)$
$-B_{(p)}(R_4, Y)$	−0.042	−0.051	−0.063	−0.050	−0.055	−0.064	−0.083	−0.064	$-B(R_4, Y)$
$B_{(p)}(X_4, Y)$	0.179	0.173	0.160	0.146	0.148	0.170	0.183	0.154	$B(X_4, Y)$
$B_{(p)}(Y^*, Y)$	1.011	1.028	1.087	1.213	1.244	1.392	1.471	1.231	$B(Y^*, Y)$
$-B_{(p)}(R, Y)$	−0.210	−0.279	−0.411	−0.580	−0.611	−0.716	−0.753	−0.553	$-B(R, Y)$
$I_{(p)}(Y)$	0.801	0.749	0.676	0.633	0.633	0.676	0.718	0.676	$I(Y)$

The last row of Table 25 shows that the behaviour of $I_{(p)}(Y)$ is U-shaped. This characteristic has been observed in many income distributions. For more details see: Zenga (2007, 2007b), Maffenini, Polisicchio (2014), Greselin, Pasquazzi, Zitikis (2010), Langel, Tillé (2012), Arcagni, Zenga (2013).

Note that the contributions $B_{(p)}\left(R_j, Y\right)$ and $B_{(p)}(R, Y)$, reported in Table 25, are greater than the corresponding contributions $B_{(p)}\left(R_j, Y^*\right)$ and $B_{(p)}(R, Y^*)$, reported in Table 23. In effect,

$$B_i\left(R_j, Y\right) = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(Y)} \geq \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(Y^*)} = B_i\left(R_j, Y^*\right) \Longrightarrow$$

$$B_i\left(R_j, Y^*\right) = B_i\left(R_j, Y\right) \cdot \frac{\overset{+}{M}_i(Y)}{\overset{+}{M}_i(Y^*)}.$$

Consequently,

$$\sum_j B_i(R_j, Y^*) = \frac{\overset{+}{M}_i(Y)}{\overset{+}{M}_i(Y^*)} \cdot \sum_j B_i(R_j, Y) \Longrightarrow B_i(R, Y^*) = \delta_i^{-1} \cdot B_i(R, Y).$$

Now we point out that in spite of the above mentioned disequalities, $\{B_i(R_j, Y) \geq B_i(R_j, Y^*)\}$, it results that:

$$\frac{B_i(R, Y)}{B_i(Y^*, Y)} = \frac{\overset{+}{M}_i(R) - \overset{-}{M}_i(R)}{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)} = \frac{B_i(R, Y^*)}{I_i(Y^*)} = \beta_i(R, Y^*); \tag{83}$$

$$\frac{B_i(R_j, Y)}{B_i(Y^*, Y)} = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)} = \frac{B_i(R_j, Y^*)}{I_i(Y^*)} = \beta_i(R_j, Y^*). \tag{84}$$

For the marginal ratios $\frac{B(R,Y)}{B(Y^*,Y)}$ and $\frac{B(R_j,Y)}{B(Y^*,Y)}$, obtainable from Table 25, we have:

$$\frac{B(R, Y)}{B(Y^*, Y)} = \frac{\frac{1}{n}\sum_i B_i(R, Y)}{\frac{1}{n}\sum_i B_i(Y^*, Y)} = \frac{\sum_i \frac{B_i(R,Y)}{B_i(Y^*,Y)} \cdot B_i(Y^*, Y)}{n \cdot B(Y^*, Y)} =$$
$$= \sum_i \beta_i(R, Y^*) \cdot \frac{B_i(Y^*, Y)}{n \cdot B(Y^*, Y)}; \tag{85}$$

$$\frac{B(R_j, Y)}{B(Y^*, Y)} = \frac{\frac{1}{n}\sum_i B_i(R_j, Y)}{\frac{1}{n}\sum_i B_i(Y^*, Y)} = \frac{\sum_i \frac{B_i(R_j,Y)}{B_i(Y^*,Y)} \cdot B_i(Y^*, Y)}{n \cdot B(Y^*, Y)} =$$
$$= \sum_i \beta_i(R_j, Y^*) \cdot \frac{B_i(Y^*, Y)}{n \cdot B(Y^*, Y)}. \tag{86}$$

TABLE 26. - *Relative contributions of the reordering variates to the contributions $B_{(p)}(Y^*, Y)$ and $B(Y^*, Y)$*

	percentiles p								
Point ratios	0.05	0.10	0.25	0.5	0.75	0.90	0.95	Marginal ratios	
$\frac{B_{(p)}(R_1, Y)}{B_{(p)}(Y^*, Y)}$	0.032	0.048	0.095	0.177	0.170	0.160	0.169	0.141	$B(R_1, Y)/B(Y^*, Y)$
$\frac{B_{(p)}(R_2, Y)}{B_{(p)}(Y^*, Y)}$	0.128	0.164	0.213	0.231	0.202	0.167	0.145	0.188	$B(R_2, Y)/B(Y^*, Y)$
$\frac{B_{(p)}(R_3, Y)}{B_{(p)}(Y^*, Y)}$	0.007	0.01	0.013	0.029	0.075	0.142	0.141	0.067	$B(R_3, Y)/B(Y^*, Y)$
$\frac{B_{(p)}(R_4, Y)}{B_{(p)}(Y^*, Y)}$	0.042	0.050	0.058	0.041	0.044	0.046	0.056	0.052	$B(R_4, Y)/B(Y^*, Y)$
$\frac{B_{(p)}(R, Y)}{B_{(p)}(Y^*, Y)}$	0.208	0.271	0.378	0.478	0.491	0.514	0.512	0.449	$B(R, Y)/B(Y^*, Y)$

According to what remarked above, the point ratios reported in Table 26 are equal to the correspondent relative contributions of the reordering variates to the index $I_i(Y^*)$ reported in Table 24. Viceversa, the values reported in the last column of these two tables are not equal. For example: $\frac{B(R,Y)}{B(Y^*,Y)} = 0.449 > 0.435 = \beta(R, Y^*)$. In the application at hand this inequality is due to the fact that the weight $B_i(Y^*, Y)/n \cdot B(Y^*, Y)$ is co-graduated w.r.t. $\beta_i(R, Y^*)$, while the weight $I_i(Y^*)/n \cdot I(Y^*)$ is contra-graduated w.r.t. $\beta_i(R, Y^*)$.

8.2 *Contribution $B(X_j, Y)$ as function of: the inequality index $I(X_{(j)})$, the contribution $B(R_j, X_{(j)})$ and the ratio $\tilde{A}_j$*

Tables 27 and 28 report respectively the contributions and the relative contributions of the variates X_j and R_j, $(j = 1, 2, 3, 4)$, to $I_{(p)}(X_{(j)})$ and to $I(X_{(j)})$.

TABLE 27. - *Contributions of the variates X_j and R_j to $I_{(p)}(X_{(j)})$ and $I(X_{(j)})$*

	percentiles p 0.05	0.10	0.25	0.50	0.75	0.90	0.95		
$B_{(p)}(R_1, X_{(1)})$	0.087	0.132	0.258	0.436	0.386	0.411	0.471	0.364	$B(R_1, X_{(1)})$
$B_{(p)}(X_1, X_{(1)})$	0.913	0.868	0.742	0.563	0.478	0.411	0.353	0.578	$B(X_1, X_{(1)})$
$I_{(p)}(X_{(1)})$	1.000	1.000	1.000	1.000	0.864	0.822	0.824	0.942	$I(X_{(1)})$
$B_{(p)}(R_2, X_{(2)})$	0.415	0.530	0.677	0.705	0.602	0.547	0.493	0.592	$B(R_2, X_{(2)})$
$B_{(p)}(X_2, X_{(2)})$	0.585	0.471	0.323	0.232	0.209	0.274	0.321	0.325	$B(X_2, X_{(2)})$
$I_{(p)}(X_{(2)})$	1.000	1.000	1.000	0.937	0.811	0.821	0.814	0.918	$I(X_{(2)})$
$B_{(p)}(R_3, X_{(3)})$	0.059	0.087	0.119	0.233	0.395	0.518	0.483	0.301	$B(R_3, X_{(3)})$
$B_{(p)}(X_3, X_{(3)})$	0.941	0.913	0.881	0.767	0.605	0.467	0.484	0.696	$B(X_3, X_{(3)})$
$I_{(p)}(X_{(3)})$	1.000	1.000	1.000	1.000	1.000	0.985	0.967	0.997	$I(X_{(3)})$
$B_{(p)}(R_4, X_{(4)})$	0.189	0.229	0.265	0.206	0.202	0.217	0.257	0.260	$B(R_4, X_{(4)})$
$B_{(p)}(X_4, X_{(4)})$	0.811	0.770	0.674	0.589	0.556	0.574	0.568	0.593	$B(X_4, X_{(4)})$
$I_{(p)}(X_{(4)})$	1.00	0.999	0.939	0.794	0.758	0.791	0.826	0.853	$I(X_{(4)})$

First of all we remark that all the four variates $X_{(j)}$ have very high synthetic indexes. In particular: $I(X_{(3)}) = 0.997$, at least 80% of the households have self-employment income equal to zero; $I(X_{(1)}) = 0.942$, at least 45% of the households have payroll income equal to zero; $I(X_{(2)}) = 0.918$, at least 35% of the households have pensions and net transfers equal to zero; $I(X_{(4)}) = 0.853$, at least 5% of the households have property income equal to zero.

TABLE 28. - *Relative contributions of the variates X_j and R_j to $I_{(p)}\left(X_{(j)}\right)$ and $I\left(X_{(j)}\right)$*

	percentiles p 0.05	0.10	0.25	0.50	0.75	0.90	0.95	
$\beta_{(p)}\left(R_1, X_{(1)}\right)$	0.087	0.132	0.258	0.436	0.447	0.500	0.572	0.386 $\beta\left(R_1, X_{(1)}\right)$
$\beta_{(p)}\left(X_1, X_{(1)}\right)$	0.913	0.868	0.742	0.564	0.553	0.500	0.428	0.614 $\beta\left(X_1, X_{(1)}\right)$
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\beta_{(p)}\left(R_2, X_{(2)}\right)$	0.415	0.529	0.677	0.753	0.742	0.666	0.605	0.645 $\beta\left(R_2, X_{(2)}\right)$
$\beta_{(p)}\left(X_2, X_{(2)}\right)$	0.588	0.471	0.323	0.247	0.258	0.334	0.395	0.354 $\beta\left(X_2, X_{(2)}\right)$
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\beta_{(p)}\left(R_3, X_{(3)}\right)$	0.059	0.087	0.119	0.233	0.395	0.526	0.500	0.302 $\beta\left(R_3, X_{(3)}\right)$
$\beta_{(p)}\left(X_3, X_{(3)}\right)$	0.941	0.913	0.881	0.767	0.605	0.474	0.500	0.698 $\beta\left(X_3, X_{(3)}\right)$
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\beta_{(p)}\left(R_4, X_{(4)}\right)$	0.189	0.229	0.283	0.259	0.267	0.274	0.312	0.305 $\beta\left(R_4, X_{(4)}\right)$
$_4\beta_{(p)}\left(X_4, X_{(4)}\right)$	0.811	0.771	0.717	0.741	0.733	0.726	0.688	0.695 $\beta\left(X_4, X_{(4)}\right)$
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

To appreciate the informations reported in Table 28 we illustrate now, for the percentile $p = 0.10$, the relative contributions of X_2 (Pensions and net transfers) and X_3 (Net self-employment income).

$$\beta_{(0.1)}\left(X_2, X_{(2)}\right) = \frac{\overset{+}{M}_{(0.1)}(X_2) - \overset{-}{M}_{(0.1)}(X_2)}{\overset{+}{M}_{(0.1)}\left(X_{(2)}\right) - \overset{-}{M}_{(0.1)}\left(X_{(2)}\right)} =$$
$$= \frac{10605.3 - 5338.3}{11198.5 - 0.0000} = \frac{5267}{11198.5} = 0.471.$$

This ratio means that, for this percentile, the difference between the upper mean and the lower mean of the ''actual'' Pension and transfer income is the 47.1% of the same difference in the case of uniform cograduation among the components. Note that in this hyphotesis the lower mean is equal to zero,while the correspondent ''actual'' mean is equal to 5338.3 euro.

$$\beta_{(0.1)}\left(X_3, X_{(3)}\right) = \frac{\overset{+}{M}_{(0.1)}(X_3) - \overset{-}{M}_{(0.1)}(X_3)}{\overset{+}{M}_{(0.1)}\left(X_{(3)}\right) - \overset{-}{M}_{(0.1)}\left(X_{(3)}\right)} =$$
$$= \frac{3978.9 - 314.4}{4013.8 - 0.000} = \frac{3664.5}{4013.8} = 0.913.$$

This latter ratio informs that for the Net self-employment income the difference between the upper mean and the lower mean of the ''actual data'' is the 91.3% of the same difference in the case of uniform cograduation. In other words the influence of the reordering variate R_3 is scarce.

We end this section with the representation (48) of $B_{(0.75)}(X_1, Y)$ and the representation (54) of $B(X_4, Y)$. Thus,

a) $B_{(0.75)}(X_1, Y) = \{I_{(0.75)}(X_{(1)}) - B_{(0.75)}(R_1, X_{(1)})\} \cdot {}_1\overset{+}{\gamma}_{(0.75)} \cdot \delta_{(0.75)} =$

$= \{0.864 - 0.386\} \cdot 0.371 \cdot 1.45 = 0.258;$

b) $B_{(0.75)}(X_1, Y) = B_{(0.75)}(X_1, X_{(1)}) \cdot {}_1\overset{+}{\gamma}_{(0.75)} \cdot \delta_{(0.75)} = 0.478 \cdot 0.371 \cdot 1.453 = 0.258;$

c) $B_{(0.75)}(X_1, Y) = B_{(0.75)}(X_1, Y^*) \cdot \delta_{(0.75)} = 0.177 \cdot 1.453 = 0.258.$

Finally, for the contribution of X_4 to $I(Y)$ we have,

a) $B(X_4, Y) = \{I(X_{(4)}) - B(R_4, X_{(4)})\}{}_4\tilde{\gamma} \cdot {}_4\tilde{\delta}$

$= \{0.853 - 0.260\} \cdot 0.204 \cdot 1.2727 = 0.154;$

b) $B(X_4, Y) = B(X_4, X_{(4)}) \cdot {}_4\tilde{\gamma} \cdot {}_4\tilde{\delta} = 0.593 \cdot 0.204 \cdot 1.2727 = 0.154;$

c) $B(X_4, Y) = B(X_4, Y^*) \cdot {}_4\tilde{\delta} = 0.121 \cdot 1.2727 = 0.154.$

TABLE 29. - *Contributions* $B(X_j, Y^*), B(X_j, X_{(j)}), B(X_j, Y), (j = 1, 2, 3, 4)$, *and means* ${}_j\tilde{\gamma}$, ${}_j\tilde{\delta}$ *and* $\tilde{A}_j$

	$B(X_j, Y^*)$	$B(X_j, X_{(1)})$	$B(X_j, Y)$	${}_j\tilde{\gamma}$	${}_j\tilde{\delta}$	$\tilde{A}_j$
j	(1)	(2)	(3)	$\frac{(1)}{(2)}$	$\frac{(3)}{(1)}$	$\frac{(3)}{(2)}$
1	0.213	0.578	0.266	0.3685	1.2488	0.4602
2	0.094	0.325	0.128	0.2892	1.3617	0.3938
3	0.096	0.696	0.127	0.1379	1.3229	0.18247
4	0.121	0.593	0.154	0.204	1.2727	0.25969
	0.524		0.676			
	$B(Y, Y^*)$		$I(Y)$			

9. CONCLUSIONS AND FINAL REMARKS

The n values of the total income Y arranged in non-decreasing order are: $y_{(1)} \leq ... \leq y_{(i)} \leq ... \leq y_{(n)}$. The synthetic inequality index $I(Y)$ is the (simple) arithmetic mean of the corresponding point indexes $I_i(Y)$: $I(Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(Y)$. The total income Y is the sum $\sum_{j=1}^{c} X_j$ of the components X_j. Obviously, $y_{(i)} = \sum_{j=1}^{c} x_{ij}$, where $x_{i1}, ..., x_{ij}, ..., x_{ic}$ are the values assumed by the c variates X_j on the population unit corresponding to $y_{(i)}$. The decomposition of the point measure is: $I_i(Y) = \sum_{j=1}^{c} B_i(X_j, Y)$, where $B_i(X_j, Y)$ is the contribution of the component

X_j to the point index. The decomposition by sources of the synthetic index is: $I(Y)=\sum_{j=1}^{c} B(X_j, Y)$, where $B(X_j, Y) = \frac{1}{n} \cdot \sum_{i=1}^{n} B_i(X_j, Y)$ is the contribution of X_j to $I(Y)$. Now, we remark that, the aforementioned approach for the decomposition by sources of $I(Y)$ can be utilized for any synthetic measure $H(Y)$ such that: $H(Y) = \frac{1}{n} \cdot \sum_i H_i(Y) = \frac{1}{n} \cdot \sum_i \left\{ \sum_j L_i(X_j) \right\}$, where $L_i(X_j)$ is the contribution of X_j to the point inequality measure $H_i(Y)$. Zenga (2013) used this approach to compare the decomposition by sources of the Gini (1914), Bonferroni (1930) and Zenga (2007) inequality indexes. Arcagni and Zenga (2014) used this approach to decompose by sources the Zenga (1984) inequality index, too.

$X_{(1)},\ldots,X_{(j)},\ldots,X_{(c)}$ are the sources (with values) arranged in non-decreasing order. The n values assumed by $X_{(j)}$ are: $x_{(1j)} \leq \ldots \leq x_{(ij)} \leq \ldots \leq x_{(nj)}$. $Y^* = \sum_{j=1}^{c} X_{(j)}$ is the total income in the case of uniform cograduation among the c sources. The n values assumed by Y^* are: $y^*_{(i)} = x_{(i1)} + \ldots + x_{(ij)} + \ldots + x_{(ic)}$. $I_i(Y^*)$ and $I(Y^*) = \frac{1}{n} \sum_i I_i(Y^*)$ are the point and the synthetic Zenga's indexes of Y^*. Moreover, $I_i(Y^*)=\sum_{j=1}^{c} B_i(X_{(j)}, Y^*)$, where $B_i(X_{(j)}, Y^*)$ is the contribution of $X_{(j)}$ to $I_i(Y^*)$, and $B(X_{(j)}, Y^*) = \frac{1}{n} \cdot \sum_{i=1}^{n} B_i(X_{(j)}, Y^*)$ is the contribution of $X_{(j)}$ to the synthetic index $I(Y^*)$. The point and the synthetic Zenga's indexes of $X_{(j)}$ are: $I_i(X_{(j)})$ and $I(X_{(j)}) = \frac{1}{n} \cdot \sum_{i=1}^{n} I_i(X_{(j)})$. Now, we remark that: $B_i(X_{(j)}, Y^*) = I_i(X_{(j)}) \cdot {}_j\overset{+}{\gamma}_i$, where ${}_j\overset{+}{\gamma}_i = \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y^*)}$ is the share of the upper mean $\overset{+}{M}_i(X_{(j)})$ on the upper mean $\overset{+}{M}_i(Y^*)$. Thus: $I_i(Y^*) = \sum_{j=1}^{c} I_i(X_{(j)}) \cdot {}_j\overset{+}{\gamma}_i$ and $I(Y^*) = \sum_{j=1}^{c} I(X_{(j)}) \cdot {}_j\overset{*}{\gamma}$, where ${}_j\overset{*}{\gamma} = \sum_{i=1}^{n} {}_j\overset{+}{\gamma}_i \cdot \frac{I_i(X_{(j)})}{n \cdot I(X_{(j)})}$. In other words, in the case of uniform cograduation, the synthetic index $I(Y^*)$ is the weighted sum of the inequalities of the components with weights ${}_j\overset{*}{\gamma}$. This result is similar to the one obtained by Rao (1969) for the Gini index, see Section 7.

Some interesting relationships between the decompositions of the inequality index of the actual total income Y and the ones of Y^* are obtained introducing the reordering variates $R_j = X_{(j)} - X_j$. The n values assumed by R_j are: $r_{ij} = x_{(ij)} - x_{ij}, \quad (i = 1, 2, \ldots, n)$. In other words, $X_{(j)} = X_j + R_j$ and $x_{(ij)} = x_{ij} + r_{ij}$. Thus, for the point inequality index of (the sum) $X_{(j)} = X_j + R_j$ we have: $I_i(X_{(j)})= B_i(X_j, X_{(j)}) + B_i(R_j, X_{(j)})$, where $B_i(X_j, X_{(j)}) = \frac{\overset{+}{M}_i(X_j) - \overset{-}{M}_i(X_j)}{\overset{+}{M}_i(X_{(j)})}$ and $B_i(R_j, X_{(j)}) = \frac{\overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j)}{\overset{+}{M}_i(X_{(j)})}$ are the contributions of X_j and R_j to $I_i(X_{(j)})$. $\overset{+}{M}_i(X_j)$ and $\overset{+}{M}_i(R_j)$ are the upper means of X_j and R_j, while $\overset{-}{M}_i(X_j)$ and $\overset{-}{M}_i(R_j)$ are the corresponding lower means. Moreover: $I(X_{(j)}) = B(X_j, X_{(j)}) + B(R_j, X_{(j)})$,where $B(X_j, X_{(j)}) = \frac{1}{n} \cdot \sum_i B_i(X_j, X_{(j)})$ and $B(R_j, X_{(j)}) = \frac{1}{n} \cdot \sum_i B_i(R_j, X_{(j)})$.Now it is worth to remark (Lemma 1, Section 5) that $\left\{ \overset{+}{M}_i(R_j) - \overset{-}{M}_i(R_j) \right\} \geq 0, \forall(i,j)$. Consequently, $B_i(R_j, X_{(j)}) \geq 0, \forall(i,j)$, and $B(R_j, X_{(j)}) \geq 0, \forall(j)$. In other words the contribution to $I\left(X_{(j)}\right)$ resulting from the reordering of X_j is greater or equal to zero, with equality only in the case $\left(x_{ij} = x_{(ij)}\right), \forall(i)$. Tables 27 and 28 shown that the ‘‘influence’’ on the inequality of the reordering variate is: scarce for the Self-employment income and relevant for the Pensions and net transfers income.

Using the relation $X_{(j)} = X_j + R_j$ we have: $Y^* = \sum_j \left(X_j + R_j\right) = Y + R$, where $R = \sum_j R_j$. Thus, for $I_i(Y^*)$ and $I(Y^*)$ we have the following $c \times 2$ bivariate decomposition:

$$I_i(Y^*) = \sum_{j=1}^{c} B_i\left(X_{(i)}, Y^*\right) = \sum_{j=1}^{c}\left(B_i\left(X_j, Y^*\right) + B_i\left(R_j Y^*\right)\right) = B_i(Y, Y^*) + B_i(R, Y^*),$$

where $B_i\left(X_j, Y^*\right) = \frac{\overset{+}{M}_i\left(X_j\right) - \overset{-}{M}_i\left(X_j\right)}{\overset{+}{M}_i(Y^*)}$, $B_i\left(R_j, Y^*\right) = \frac{\overset{+}{M}_i\left(R_j\right) - \overset{-}{M}_i\left(R_j\right)}{\overset{+}{M}_i(Y^*)}$, $B_i(Y, Y^*) = \frac{\overset{+}{M}_i(Y) - \overset{-}{M}_i(Y)}{\overset{+}{M}_i(Y^*)}$ and $B_i(R, Y^*) = \frac{\overset{+}{M}_i(R) - \overset{-}{M}_i(R)}{\overset{+}{M}_i(Y^*)}$;

$$I(Y^*) = \sum_{j=1}^{c} B\left(X_{(j)}, Y^*\right) = \sum_{j=1}^{c}\left(B\left(X_j, Y^*\right) + B\left(R_j, Y^*\right)\right) = B(Y, Y^*) + B(R, Y^*).$$

The last column of Table 23 reports the 4×2 decomposition of $I(Y^*)$. In particular, the contribution of the total reordering variate is $B(R, Y^*) = 0.403$; this latter value is the 43.5% of the synthetic index $I(Y^*)$=0.927. In other words the value of $I(Y^*)$ is the sum of the contribution of the actual total income Y (56.5%) and of the contribution of the total reordering variate R (43.5%).

Using the relation $X_j = X_{(j)} - R_j$ we have $Y = \sum_j \left(X_{(j)} - R_j\right) = Y^* - R$. Thus:

$$I_i(Y) = \sum_{j=1}^{c} B_i\left(X_j, Y\right) = \sum_{j=1}^{c}\left(B_i\left(X_{(j)}, Y\right) - B_i\left(R_j, Y\right)\right) = Bi(Y^*, Y) - B_i(R, Y),$$

where: $B_i\left(X_{(j)}, Y\right) = \frac{\overset{+}{M}_i\left(X_{(j)}\right) - \overset{-}{M}_i\left(X_{(j)}\right)}{\overset{+}{M}_i(Y)}$, $B_i\left(R_j, Y\right) = \frac{\overset{+}{M}_i\left(R_j\right) - \overset{-}{M}_i\left(R_j\right)}{\overset{+}{M}_i(Y)}$, $B_i(Y^*, Y) = \frac{\overset{+}{M}_i(Y^*) - \overset{-}{M}_i(Y^*)}{\overset{+}{M}_i(Y)}$, and $B_i(R, Y) = \frac{\overset{+}{M}_i(R) - \overset{-}{M}_i(R)}{\overset{+}{M}_i(Y)}$;

$$I(Y) = \sum_{j} B\left(X_j, Y\right) = \sum_{j}\left(B\left(X_{(j)}, Y\right) - B\left(R_j, Y\right)\right) = B(Y^*, Y) - B(R, Y).$$

For the application at hand Table 28 reports these latter decompositions. It is worth to note that $B_{(p)}(Y^*, Y)$ and $B(Y^*, Y)$ are greater than 1. In effect, $B_{(p)}(Y^*, Y) = I_{(p)}(Y^*) \cdot \frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)} = I_{(p)}(Y^*) \cdot \delta_{(p)}$, where: $\delta_{(p)}$is an increasing function greater than 1, $I^*_{(p)}(Y^*) \simeq 1$ for $p \leq 0.25$, and decreases slowly for $p > 0.25$. Many interesting informations can be obtained from Table 25. For example the contribution $B_{(0.75)}(X_1, Y) = 0.258$ is obtained subtracting the contribution $B_{(0.75)}(R_1, Y) = 0.211$ (of the reordering variate R_1) to the contribution $B_{(0.75)}\left(X_{(1)}, Y\right) = 0.496$ (of the ordered variate $X_{(1)}$).

Finally, we can show that $B_i(X_j, Y)$ is function of $I_i(X_{(j)})$, of the contribution $B_i(R_j, X_{(j)})$, and of the ratios ${}_j\overset{+}{\gamma}_i$ and δ_i. In effect:

$$\begin{aligned} B_i(X_j, Y) &= \{B_i(X_{(j)}, Y) - B_i(R_j, Y)\} \cdot \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(X_{(j)})} \\ &= \{I_i(X_{(j)}) - B_i(R_j, X_{(j)})\} \cdot \frac{\overset{+}{M}_i(X_{(j)})}{\overset{+}{M}_i(Y)} = \\ &= \{I_i(X_{(j)}) - B_i(R_j, X_{(j)})\} \cdot {}_j\overset{+}{\gamma}_i \end{aligned}$$

In other words the reordering variate R_j decreases the influence of the inequality index $I_i(X_{(j)})$ on $B_i(X_j, Y)$, while all the reordering variates increase the contribution $B_i(X_j, Y)$ because $\frac{\overset{+}{M}_i(Y^*)}{\overset{+}{M}_i(Y)} = \left(1 + \frac{\overset{+}{M}_i(R)}{\overset{+}{M}_i(Y)}\right) \geq 1$. Note that the share ${}_j\overset{+}{\gamma}_i$ is not influenced by the reordering variates. The aforementioned representation of $B_i(X_j, Y)$ can be extended to the contribution $B(X_j, Y)$ of the synthetic index $I(Y)$: $B(X_j, Y) = \{I(X_{(j)}) - B(R_j, X_{(j)})\} \cdot {}_j\tilde{\gamma} \cdot {}_j\tilde{\delta}$, where: ${}_j\tilde{\gamma} = \frac{B(X_j, Y^*)}{B(X_j, X_{(j)})}$ and ${}_j\tilde{\delta} = \frac{B(X_j, Y)}{B(X_j, Y^*)}$. It is worth to remark that ${}_j\tilde{\gamma}$ and ${}_j\tilde{\delta}$ are ''weighted'' means of the corresponding ratios ${}_j\overset{+}{\gamma}_i$ and δ_i.

In this paper we have also compared the three decompositions by sources of the synthetic Gini index $G(Y)$ proposed by Rao (1969), by Lerman *et al.* (1984, 1985) and by Radaelli *et al.* (2005). This latter decomposition uses the reordering variates and, what is more important, its interpretation seems to be clearer than those of the other two decompositions.

Mussini and Zenga (2013) investigated the role of reordering in determining inequality changes between two points in time. They proposed a decomposition of the Zenga inequality index which isolates the contributions of reordering and income growth to the change in inequality occurred from an initial time to a final time.

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