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On the completeness of the family of factorizable densities

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Summary: *The family of continuous distribution laws which factorize in two positive functions, one depending on the variable only and the other one depending on a scalar parameter exclusively, is concerned. The family has no intersection with the Exponential class and includes Uniform distributions over an interval depending upon the parameter as well as infinitely many other distributions arising, for instance, by truncation. Simple sufficient conditions, regarding the support of random variables with distribution included in the family of factorizable densities and suitable to prove completeness are stated. Completeness of statistics based on sample from densities included in the family is also considered.*

Keywords: *Acceptance-Rejection method, Complete statistics, Non-regular distributions families, Truncated data.*

1. Introduction

Despite its unintuitive statistical meaning, the property of completeness of a parametric family of distributions plays a significant role in the theory of inference.

We recall, for instance, Basu's theorem stating the independency of a sufficient and complete statistic S from any ancillary statistic (e.g. Shao, 1999, p. 82). As a consequence the property of completeness in some sense completes the more intuitive property of sufficiency : given a sufficient statistic S in order to exclude that some function $f(S)$ could be ancillary - i.e. totally uninformative with regard to the parameter - also completeness is needed. On the other hand, an ancillary statistic can not be a function of a

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complete statistic. Moreover, a sufficient and complete statistic is also minimal sufficient as follows from known results by Lehmann and Scheffé (1950) and by Bahadur (1957). Hence the property of completeness supplements the sufficiency by implying also minimality - i.e. the maximum attainable synthesis of sample data. Particularly with regard to an unbiased estimator, if T is a function of a sufficient and complete statistic then T has uniformly minimum variance - i.e. it is the most efficient estimator in the class of unbiased estimators for a given parameter. Notice also that completeness confers a sort of uniqueness upon an estimator since, given a complete statistic S, only a function of S can have an assigned expected value. Thus, if a function $T(S)$ is an unbiased estimator no other function of S will be (e.g. Stuart and Ord, 1999, p. 201).

Finally, completeness can be an useful tool in testing statistical hypotheses according to Neyman-Pearson approach (e.g. Lehmann, 1986, p. 144).

On the other hand, checking completeness of a family of distributions by applying the definition is usually not an easy task.

In this note simply sufficient conditions are given in order to ensure completeness of a particular class of distribution laws. Conditions regard essentially the support of random variables with distribution laws included in the family and few results concerning the family we will focus on already exist in the literature.

Particularly, this note deals with the class F of distribution of one dimensional continuous random variable X with the following form

$$F = \{\varphi(x;\theta) = h(x)k(\theta)1_{\Omega_\theta}(x), \quad x \in \Omega_\theta \subseteq \mathbf{R}, \quad \theta \in \Theta \subseteq \mathbf{R}\} \qquad (1)$$

where the parameter space Θ is an interval of the real line $\mathbf{R}$ whose endpoints can be finite or infinite and can belong or not to Θ; Ω_θ is the support of X and 1_{Ω_θ} is its indicator function, h is a strictly positive Lebesgue measurable function which does not depend on θ and k is a strictly positive function defined on Θ which does not depend on x.

We call F family of factorizable densities.

Whereas φ is a density and h does not depend on θ, $\int_{\Omega_\theta} \varphi(x;\theta)dx = 1$ implies $\int_{\Omega_\theta} h(x)dx = 1/k(\theta)$, hence the support Ω_θ does depend on θ as the notation Ω_θ emphasizes; we assume Ω_θ is an interval

$$\Omega_\theta = \{x : a(\theta) < x < b(\theta)\} \qquad (2)$$

where a and b are monotone functions defined on the parameter space Θ.

Notice that the family F has no intersection with the Exponential family since the support depends on the parameter. Owing to the same reasons, the family F has been considered in the literature as a $\text{non}-\text{regular case}$ (e.g. Hogg and Craig, 1956).

Furthermore, the family F includes Uniform densities over Ω_θ as well as infinitely many other densities for which to prove completeness by applying the definition could result in a tricky task. As a consequence, the family F appears sufficiently wide to be interesting under theoretical aspects.

From a more operative point of view, notice that factorizable densities arise from usual distributions by arranging some truncation. For instance, let X be the random variable interpreting the duration of a certain electronic component with minimum duration θ. Suppose a suitable model is the Negative Exponential distribution with unitary parameter if $\theta = 0$. If $\theta > 0$ has to be assumed then $\varphi(x;\theta) = \exp(\theta - x)1_{(\theta,\infty)}(x)$ which is included in the family defined by (1). As another example, let X be the random variable with density $\varphi(x;\theta) = x\exp(-x)/\{1-\exp[-\theta(1+\theta)]\}1_{(0,\theta)}(x)$. Hence $\varphi \in F$ and X has a Gamma distribution with parameters 2 and 1 truncated on the right at $\theta > 0$.

Furthermore, an interesting application of the family F arises in the context of stochastic simulations. If X has a density included in the class F defined by (1), it is easy to verify that if $H(x)$ is a primitive function of $h(x)$, then the random variable $H(X)$ has uniform distribution on the interval $(H(a(\theta)), H(b(\theta)))$. Consequently, by resorting to this property, the family (1) represents a class of densities from which it is easy to generate pseudo-values by the Acceptance-Rejection method (e.g. Rubinstein, 1981, p. 45).

EXAMPLE 1

Let

$$\varphi(x;\theta) = (5x^4 + 1)/(\theta^5 + \theta)1_{(0,\theta)}(x), \quad \theta > 0$$

be the density of a random variable X. Assume we are interested in generating pseudo-values from X. The cumulative distribution $F(x;\theta) = \int_0^x \varphi(t;\theta)dt = (x^5 + x)/(\theta^5 + \theta)$ although monotonic has inverse function which is regular but it is not an elementary function, i.e. it can not be expressed analytically in terms of algebraic, trigonometric or exponential functions). Consequently it is not possible to generate exactly from X by means of the Inverse Transform method (e.g. Rubinstein, 1981, p. 39).

Nevertheless, since $\varphi(x;\theta)$ is a member of the family F with $h(x)=5x^4+1$, the random variable $H(X)=X^5+X$ has Uniform distribution on the interval $(0,\theta^5+\theta)$. Thus it is simple to generate from X with the Acceptance-Rejection method by using the following decomposition

$$\varphi(x;\theta)= f(x)g(x)C=\frac{1}{\theta^5+\theta}\cdot\frac{5x^4+1}{5\theta^4+1}\cdot(5\theta^4+1)$$

where, as requested by method, f is a density function easy to generate from, g is a positive function taking value on the interval $(0,1)$ and C is a real constant greater or equal than 1.

In this paper completeness of the family F, as recalled in Definition 1, is concerned.

DEFINITION 1

The family F is complete if for any real valued function g locally integrable with respect to the measure $h(x)dx$ on the set $\Omega=\bigcup_{\theta\in\Theta}\Omega_\theta$, the vanishing of

$$E_\theta[g(X)]=\int_{\Omega_\theta} g(x)h(x)k(\theta)dx,\quad \forall\theta\in\Theta \qquad (3)$$

implies $g=0$ almost surely (a.s.) on Ω.

Let $a(\theta)$ and $b(\theta)$ be the functions appearing in (2). In order to state sufficient conditions concerning completeness of the family F, three cases will be distinguished :

i) $a(\theta)$, or similarly $b(\theta)$, is constant;

ii) $a(\theta)$ is strictly monotone decreasing and $b(\theta)$ is strictly monotone increasing (or vice-versa);

iii) $a(\theta)$ and $b(\theta)$ are strictly monotone increasing (decreasing).

In Section 2 it will be shown that, under quite general conditions, the family F is complete if case i) holds (Theorem 1) and it is not complete in the remaining cases (Theorems 2 and 3). Completeness of statistics based on samples from densities included in the family F is also considered. It is known that for cases i) and ii) there exists a single sufficient statistic S (e.g. Stuart et al, 1999, p. 33) and S is also complete (e.g. Hogg and Craig, 1956). In case iii) it does not exist a single sufficient statistic (e.g. Stuart et al, 1999, p. 34) and no results concerning completeness have appeared in the literature. Nevertheless the pair of extreme observations $(X_{(1)},X_{(n)})$ is jointly sufficient. In Section 3 the gap will be filled by showing that $(X_{(1)}, X_{(n)})$ is not jointly complete (Theorem 4).

2. Completeness of F

To start with we observe that F can not be complete if $\alpha = \sup_{\theta\in\Theta} a(\theta) < \beta = \inf_{\theta\in\Theta} b(\theta)$. Indeed, when the strict inequality holds, all the intervals Ω_θ contain the interval $[\alpha, \beta]$. As a consequence, any function g with support included in $[\alpha, \beta]$ and such that $\int_\alpha^\beta g(x)h(x)dx = 0$, for instance

$$g(x) = \sin(\frac{2\pi(x-\alpha)}{\beta-\alpha})/h(x)1_{[\alpha,\beta]}(x)$$

shows that F is not complete. Henceforth we will assume $\alpha \geq \beta$.

Furthermore, noticed that the parameter space Θ reduces to the interval $(0,1)$, possibly closed or half closed, by reparametrization i.e. by using a suitable diffeomorphism so that the function b results increasing. For instance if $\Theta = (0,\infty)$ with $a(\theta) = 1/\theta$ and $b(\theta) = 2/\theta$, the diffeomorphism $\theta' = 2\arctan(1/\theta)/\pi$ deals with $\Theta' = (0,1)$ and with the increasing functions $a_1(\theta') = \tan(\pi\theta'/2)$ and $b_1(\theta') = 2\tan(\pi\theta'/2)$. Henceforth we shall assume the parameter space $\Theta = (0,1)$.

THEOREM 1
If $a(\theta) = \alpha$ and if b is continuous and increasing then the family F is complete.

PROOF
Let g be integrable with respect to $h(x)dx$ on all sets Ω_θ and such that :

$$E_\theta[g(X)] = k(\theta)\int_\alpha^{b(\theta)} g(x)h(x)dx = 0, \quad \forall\theta\in\Theta$$

Then the absolutely continuous function $F(x) = \int_\alpha^x g(t)h(t)dt$ vanishes on the interval $\Omega = (\alpha, \sup_{\theta\in\Theta} b(\theta))$ since, for any $x \in \Omega$, there exists $\theta \in \Theta$ such that $x = b(\theta)$. Thus $g = 0$ a.s. on Ω.◆

EXAMPLE 2 (Casella and Berger, 1990, p. 261) Let

$$F = \{\varphi(x;\theta) = 1/\theta, \quad 0 < x < \theta, \quad \theta > 0\}$$

be the family of Uniform distributions on the interval $\Omega_\theta = (0,\theta)$. Since $a(\theta) = 0$ is constant and $b(\theta) = \theta$, $\theta > 0$, is continuous and increasing, F is complete according to Theorem 1. In fact, let $g(x)$ be a real function not depending on θ and such that $E_\theta[g(X)] = 1/\theta \int_0^\theta g(x)dx = 0$. Thus $\int_0^\theta g(x)dx = 0$ and taking derivatives of both members yields $g(\theta) = 0$, for any $\theta > 0$. Hence F is complete.

THEOREM 2
Let $a(\theta)$ and $b(\theta)$ be continuous functions whose derivatives are continuous (i.e. $C^{(1)}$- functions), the former with strictly negative derivative and the latter with strictly positive derivative. Then the family F is not complete.

PROOF
Let $A = a(\Theta)$ and $B = b(\Theta)$. According to the remark at the beginning of the present Section it can be assumed A and B do not intersect; thus let $\sup A = \inf B = 0$, i.e. A and B are disjoint sets separated by 0 . A not a.s. zero function g with $E_\theta[g(X)] = 0$, $\forall \theta \in \Theta$ can be constructed as follows.
Let $\psi : B \to A$ be the decreasing function $\psi = a(b^{-1})$: Notice that ψ is a $C^{(1)}$-function. Define

$$g(x) = \begin{cases} -1/h(x) & \text{for } x \in A \\ -\psi'(x)/h(x) & \text{for } x \in B \end{cases}$$

Then

$$E_\theta[g(X)] \propto \int_{a(\theta)}^{b(\theta)} g(t)h(t)dt = -\int_{a(\theta)}^{0} dt - \int_{0}^{b(\theta)} \psi'(t)dt$$

$$= a(\theta) - \psi(b(\theta)) = 0, \ \forall \theta \in \Theta \ ◆$$

EXAMPLE 3 Let

$$F = \{\varphi(x;\theta) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}\,(\Phi(2\theta) + \Phi(\theta) - 1)}, \quad -\theta < x < 2\theta\}$$

where Φ is the cumulative distribution function of the Standard Normal distribution $N(0,1)$. Notice that F is the family of $N(0,1)$ distribution truncated in the interval $(-\theta, 2\theta)$. Since $a(\theta) = -\theta$ has strictly negative derivative and $b(\theta) = 2\theta$ has strictly positive derivative, hence F is not complete according to Theorem 2. In fact, as $h(x) = \exp(-x^2/2)$, $A = (-1,0)$, $B = (0,2)$ and $\psi(x) = -x/2$ then

$$g(x) = \begin{cases} -\exp\left(x^2/2\right) & -1 < x < 0 \\ \dfrac{1}{2}\exp\left(x^2/2\right) & 0 < x < 2 \end{cases}$$

is not a.s. zero and $E_\theta\left[g(X)\right] = 0$, $\forall \theta \in \Theta$. This shows *F* is not complete.

THEOREM 3

Let $a(\theta)$ and $b(\theta)$ be $C^{(1)}$-functions both with strictly positive derivative. Then the family F is not complete.

PROOF

Notice that if $b(\theta)$ is a translation of $a(\theta)$, i.e. $b(\theta) = a(\theta) + c$ where c is a positive real constant, then the result follows for instance by simply choosing $g(x) = \sin(2\pi x/c)/h(x)$.

In the general case assume first that Θ is open and let $a(\Theta) = (\alpha, \beta) = A$ and $b(\Theta) = (\gamma, \delta) = B$ with $\gamma < \beta$.

The function ψ defined by $\psi(x) = b(a^{-1}(x))$ is a $C^{(1)}$ bijection of A onto B with strictly positive derivative. Notice that when $x \to \alpha$ then $\psi(x) \to \gamma$ and when $x \to \beta$ then $\psi(x) \to \delta$; moreover, $\psi(x) = b(a^{-1}(x)) > a(a^{-1}(x)) = x$, $\forall x \in A$.

Choose $x_0 \in A \bigcap B$. We can construct $x_1 = \psi(x_0)$ and iteratively $x_j = \psi(x_{j-1})$ provided that $x_{j-1} \in A$. In the same way we can construct $x_{-1} = \psi^{-1}(x_0)$ and iteratively $x_{-j} = \psi(x_{-j+1})$ provided that $x_{-j+1} \in B$.

Since a is strictly increasing there exists the sequence $\{\theta_j\}_{j \in \mathbf{Z}}$, where $\mathbf{Z}$ is the set of the integers, such that $a(\theta_j) = x_j$. Notice that $b(\theta_j) = b(a^{-1}(x_j)) = \psi(x_j) = x_{j+1}$.

Since $\psi(x) > x$, $\forall x \in A$, the sequence $\{x_j\}_{j\in\mathbf{N}}$, where $\mathbf{N}$ denotes the set of the natural numbers, is strictly increasing and only two cases are possible:

i) for a certain $N \geq 0$, $x_N < \beta$ and $x_{N+1} > \beta$ so that the iteration stops and the sequence has only a finite number of terms, the last one satisfying $\beta < x_{N+1} < \delta$;

ii) the sequence $\{x_j\}_{j\in\mathbf{N}}$ has infinitely many terms and $\lim_{j\to\infty} x_j = \lim_{j\to\infty} a(\theta_j) = \lim_{j\to\infty} b(\theta_j)$ must be β forcing $\beta = \delta$.

Similarly, for negative indices j, the sequence $\{x_j\}_{j\in\mathbf{N}}$ can have either finite many or infinitely many terms; in the last case it must converge to γ and force $\gamma = \alpha$.

Let I_j denote the interval $(x_j, x_{j+1}) = (a(\theta_j), a(\theta_{j+1}))$. Note that if there are only finitely many positive indices j, the last interval I_N has the form $(x_N, \beta) = (a(\theta_N), \beta)$. Similarly, if there are only finitely many negative indices j, the first interval I_{-M} has the form $(\alpha, x_{1-M}) = (\alpha, a(\theta_{1-M}))$.

Notice that if there is a first interval I_{-M}, then $\psi(I_{-M}) = (\gamma, b(\theta_{1-M}))$, on the other hand if there is a last interval I_N, then $\psi(I_N) = (b(\theta_N), \delta))$. Hence $\psi(I_j) = I_{j+1}$

The collection of the intervals I_j, added with $\psi(I_N)$ when $\beta < \delta$, covers Ω. Distinct intervals intersect at most at their endpoints.

The function g is constructed by defining a suitable g_0 on I_0 and by using, iteratively, the diffeomorphisms ψ and ψ^{-1} to extend to the other intervals I_j.

Let g_0 be compactly supported in I_0 and such that $f_0 = g_0 h$ is continuous in I_0 and $\int_{I_0} f_0(t)dt = 0$. Iteratively, define $g_j = f_j/h$ where

$$f_j(x) = f_{j-1}(\psi^{-1}(x))(\psi^{-1})'(x) \tag{5}$$

For negative indices j, simply replace ψ^{-1} by ψ in (5). Function g is then defined on Ω by setting $g(x) = g_j(x)$ for $x \in I_j$.

When $\theta = \theta_j$, a trivial substitution gives

$$E_\theta[g(X)] \propto \int_{x_0}^{x_1} g(t)h(t)dt = \int_{I_0} f_0(t)dt = 0$$

In the general case, $\theta \in \Theta$ belongs to exactly one of the intervals (θ_j, θ_{j+1}) and $(a(\theta), b(\theta)) \subseteq (I_j \cup I_{j+1})$ with

$$a(\theta_j) = x_j < a(\theta) < a(\theta_{j+1}) = b(\theta_j) = x_{j+1} < b(\theta) < b(\theta_{j+1}) = x_{j+2}.$$

Thus

$$E_\theta[g(X)] \propto \int_{a(\theta)}^{x_{j+1}} f_j(t)dt + \int_{x_{j+1}}^{b(\theta)} f_{j+1}(t)dt. \qquad (6)$$

By using (5), since $\psi^{-1}(b(\theta)) = a(\theta)$, the substitution $s = \psi^{-1}(t)$ transforms the second integral on the right hand of (6) into $\int_{x_j}^{a(\theta)} f_j(t)dt$. This yields

$$E_\theta[g(X)] \propto \int_{I_j} f_j(t)dt = 0$$

We have assumed Θ open. If it is not open and it contains one or both of its endpoints we have $\alpha < \gamma$ or, respectively, $\beta < \delta$. This implies that there are only finitely many intervals I_j with negative (positive) index. The proof proceed now without any change. ◆

EXAMPLE 4 Let

$$F = \{\varphi(x;\theta) = \frac{\exp(\theta - x)}{1-\exp(-\theta)}, \quad \theta < x < 2\theta, \quad \theta > 0\}$$

Since $a(\theta) = \theta$ and $b(\theta) = 2\theta$ have both strictly positive derivative then F is not complete according to Theorem 3. In fact, as remarked at the beginning of this Section, we can resort to the reparametrization $\theta' = 2\arctan(1/\theta)/\pi$. Hence we can rewrite F as follows

$$F = \left\{ \varphi(x,\theta') = \frac{\exp(\tan(\theta'\pi/2) - x)}{1-\exp(-\tan(\theta'\pi/2))},\ \tan(\theta'\pi/2) < x < 2\tan(\theta'\pi/2), \right.$$

$$\left. 0 < \theta' < 1 \right\}$$

Assume $x_0 > 0$ fixed. Since $\psi(x) = 2x$ we have $I_j = (2^j x_0, 2^{j+1} x_0)$, $j \in \mathbf{Z}$. A suitable $f_0 : I_0 \to \mathbf{R}$ is then

$$f_0(x) = \sin \frac{4\pi}{x_0}(x - \frac{5}{4}x_0) 1_{[\frac{5}{4}x_0, \frac{7}{4}x_0]}(x)$$

this implies $g_0(x) = f_0(x)\exp(x)$ has compact support and $\int_{I_0} f_0(x)dx = 0$. By iterating $j \in \mathbf{Z}$ we have $f_j : I_j \to \mathbf{R}$ such that

$$f_j(x) = (\frac{1}{2})^j \sin \frac{4\pi}{x_0}(\frac{x}{2^j} - \frac{5}{4}x_0) 1_{[\frac{5}{4}2^j x_0, \frac{7}{4}2^j x_0]}(x) \quad j \in \mathbf{Z}$$

with $\int_{I_j} f_j(x)dx = 0$ and $g_j(x) = f_j(x)\exp(x)$. Thus $f(x) = f_j(x)$, $x \in I_j$, $j \in \mathbf{Z}$ and

$$E_{\theta'}[g(X)] \propto \sum_{j \in \mathbf{Z}} \int_{I_j} f_j(t)dt = 0.$$

This shows F is not complete.

3. Completeness of $(X_{(1)}, X_{(n)})$

Completeness of statistics, based on samples of size n from a density belonging to factorizable family F defined by (1), is now concerned.

As recalled in Section 1, when cases i) and ii) hold, under the condition of Theorems 1, there exists a single sufficient statistic which is also complete. If case iii) holds, under the conditions of Theorem 3 it does not exist a single sufficient statistic and we can only rely on the pair $(X_{(1)}, X_{(n)})$ which is jointly sufficient.

Since the definition of completeness for a single statistic naturally extends to jointly completeness of a pair statistic (e.g. Mood et al, 1974, p. 354), it will be now proved that $(X_{(1)}, X_{(n)})$ is not complete.

THEOREM 4

Let $a(\theta) < b(\theta)$ *be infinitely many times derivable functions (i.e.* C^{∞}*-functions) defined on* Θ *with strictly positive (negative) first derivatives and let* $h(x)$ *be a* C^{∞}*-function defined on* Ω*. Then the sufficient statistic* $(X_{(1)}, X_{(n)})$ *is not complete.*

PROOF

Without loss of generality, let assume a and b increasing. Furthermore, with the purpose of simplifying the proof, let resort to a property of F already outlined in Section 1. If $H'(x) = h(x)$, then $Y = H(X)$ has uniform distribution on $(H(a(\theta)), H(b(\theta))) = (\alpha(\theta), \beta(\theta))$. Since H is monotonic, the sufficient statistic $(X_{(1)}, X_{(n)})$ is complete if and only if the sufficient statistic $(Y_{(1)}, Y_{(n)})$ is complete.

Hence, we assume the joint density of $(X_{(1)}, X_{(n)})$ is

$$\varphi_{(X_{(1)},X_{(n)})}(x,y) = n(n-1)\frac{1}{(\beta(\theta)-\alpha(\theta))^n}(y-x)^{n-2}1_{(\alpha(\theta),\beta(\theta))}(y)1_{(\alpha(\theta),y)}(x)$$

Theorem 4 holds if we find a function $g(x,y)$ not a.s. zero, locally integrable for $y > x$ and such that

$$E_\theta[g(X_{(1)}, X_{(n)})] \propto \int_{\alpha(\theta)}^{\beta(\theta)} (\int_{\alpha(\theta)}^{y} \quad (y-x)^{(n-2)} g(x,y)dx)dy = 0, \quad \forall \theta \in \Theta$$

To start with, note that if $\beta(\theta) = \alpha(\theta) + c$, where c is a positive constant, the function

$$g(x,y) = (y-x)^{2-n} \sin^{2m}(\frac{2\pi}{c}(y-x))\cos(\frac{2\pi}{c}(y-x))$$

with $2m \geq n-2$, is not an a.s. zero function, it satisfies $E_\theta[g(X_{(1)}, X_{(n)})] = 0$, $\forall \theta \in \Theta$, and it is even bounded. Indeed, letting $y = \alpha(\theta) + t$ and $x = \alpha(\theta) + s$, yields

$$E_\theta[g(X_{(1)}, X_{(n)})] \propto \int_0^c \{\int_0^t \sin^{2m}(\frac{2\pi}{c}(t-s))\cos(\frac{2\pi}{c}(t-s))ds\}dt = 0$$

Notice that no regularity for $\alpha(\theta)$ is required. More generally, if $\inf_{\theta\in\Theta}(\beta(\theta)-\alpha(\theta))$ is greater or equal than a positive constant γ, then by choosing $g(x,y)=\phi(y-x)$ where ϕ has support $[0,\gamma]$ and it satisfies $\int_0^\gamma[\int_0^t s^{n-2}\phi(s)ds]dt=0$, the same argument used above leads to $E_\theta[g(X_{(1)},X_{(n)})]=0$. Again, no assumptions upon regularity of $\alpha(\theta)$ and $\beta(\theta)$ are needed.

In the general case and without loss of generality, let assume $\alpha(\theta)=\theta$. Set

$$g(x,y)=(y-x)^{2-n}f(x)1_{(0,y)}(x) \qquad (7)$$

and $F(y)=\int_{\theta_0}^y f(x)dx$ where θ_0 is a fixed point of Θ and f is a function with at most step discontinuities on a discrete subset of Ω. Then the vanishing of $E_\theta[g(X_{(1)},X_{(n)})]$, $\forall\theta\in\Theta$, can be written as

$$\int_\theta^{\beta(\theta)}[F(y)-F(\theta)]dy=0 \quad \text{or equivalently as}$$

$$[\beta(\theta)-\theta]F(\theta)=\int_\theta^{\beta(\theta)}F(y)dy$$

By differentiating with respect to θ leads to

$$(\beta(\theta)-\theta)F'(\theta)+\beta'(\theta)F(\theta)=\beta'(\theta)F(\beta(\theta)) \qquad (8)$$

at all continuity points of f.

Choose $\theta_0\in\Theta$ and iteratively compute $\theta_j=\beta(\theta_{j-1})$ and $\theta_{-j}=\beta^{-1}(\theta_{-j+1})$, $j\in\mathbf{N}$. It can result both finitely or infinitely many terms.

Since $\beta(\theta)>\theta$, $\forall\theta\in\Theta$, then Ω is the union of the intervals $I_j=[\theta_j,\theta_{j+1}]$ whose interiors are disjoint and such that $b(I_j)=I_{j+1}$. Little care has to be used in defining intervals I_j at the endpoints of Θ. For $\theta\in I_0$, let F be a C^∞-function with compact support K included in the interior of I_0. By using (8), since $\beta(K),\beta(\beta(K)),\beta(\beta(\beta(K))),\ldots$ are compact sets included in the interiors of the intervals $I_1,I_2,I_3,\ldots$

respectively, we can extend the definition of $F(\theta)$ to the set of all intervals I_j with positive index j. Notice that the restriction of F to I_j is a C^∞-function with compact support included in the interior of I_j.

Since $(\beta(\theta)-\theta)>0$, we can extend F to the collection of the intervals I_j with negative index j by solving a set of first order differential equations on I_j with assigned value of the unknown function $F(\theta)$ at the right end point of I_j, i.e. a set of first order Cauchy problems defined by equation (8).

Notice that such an extension is C^∞ in the interior of each I_j but it is, in general, only continuous at the points $\theta_0, \theta_{-1}, \theta_{-2}, \ldots$. However this is enough to produce f as derivative of F with step discontinuities on a discrete set only. When f is located, by substituting in equation (7), g is also derived and Theorem is then proved. ◆

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