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JOINT DECOMPOSITION BY SUBPOPULATIONS AND SOURCES OF THE ZENGA INEQUALITY INDEX $I(Y)$

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SUMMARY

The total income Y is the sum of c components $X_j : Y = \sum_{j=1}^{c} X_j$. *The N units of the population are partitioned in k different subpopulations. The Zenga point inequality index (Zenga 2007) is given by* $I_h(Y) = (\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y))/\overset{+}{M}_{h.}(Y)$ *where* $\overset{+}{M}_{h.}(Y)$ *and* $\bar{M}_{h.}(Y)$ *are the upper and lower means of Y. The synthetic index* $I(Y)$ *is the arithmetic mean* $M(\cdot)$ *of* $I_h(Y) : I(Y) = M(I_h(Y))$. *For* $I_h(Y)$ *Zenga (2016) has obtained the following decomposition by subpopulations* $I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \left[\frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}\right] p(l|h) \cdot a(g|h) = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{hlg}(Y)$, *where* $\overset{+}{M}_{hg}(Y)$ *and* $\bar{M}_{hl}(Y)$ *are the upper and lower means in the subpopulations g and l, and* $a(g|h)$ *and* p(l|h) *are their relative frequencies. Using the relations* $\bar{M}_{hg}(Y) = \sum_{j=1}^{c} \bar{M}_{hg}(X_j)$ *and* $\overset{+}{M}_{hg}(Y) = \sum_{j=1}^{c} \overset{+}{M}_{hg}(X_j)$ *in the decomposition above reported, the present paper obtains the following joint decomposition by subpopulations and sources:* $I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{hlg}(X_j)$; $B_{hlg}(X_j) = \left[\frac{\overset{+}{M}_{hg}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)}\right] p(l|h) \cdot a(g|h)$. *Finally, substituting this latter decomposition in* $I(Y) = M(I_h(Y))$,*the following* $k \times k \times c$ *joint decomposition for* $I(Y)$ *is achieved:* $I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{.lg}(X_j)$; $B_{.lg}(X_j) = M(B_{hlg}(X_j))$. *From the reported decompositions, with simple algebra, this paper shows how to obtain: the decomposition* $I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y)$, *where* $B_{hl.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(l|h)$ *can be interpreted as the contribution of the subpopulation l to the point index* $I_h(Y)$; *the decomposition* $I(Y) = \sum_{l=1}^{k}$ B.l.(Y), *where* $B_{.l.}(Y) = M(B_{hl.}(Y))$ *can be interpreted as the contribution of the subpopulation l to* $I(Y)$; *the decomposition of* $B_{hl.}(Y)$, *of* $I_h(Y)$, *of* $B_{.l.}(Y)$, *of* $I(Y)$, *and of* $B_{hl.}(X_j)$ *in a within and a between part; the shares* $\beta_{hl.}(X_j) = B_{hl.}(X_j)/B_{hl.}(Y)$, *and so on. The theoretical results of this paper are applied to the 2012 Bank of Italy sample survey on household income and wealth.*

Keywords*: Zenga Inequality Index, Income Inequality, Joint Decomposition by Subpopulations and Sources, Point and Synthetic Inequality Indexes.*

1. INTRODUCTION

The point inequality index $I_h(Y)$, proposed by Zenga (2007) is the relative variation of the arithmetic mean $\bar{M}_{h.}(Y)$ of the lower group $(Y \leq y_h)$ w.r.t. the arithmetic mean $\overset{+}{M}_{h.}(Y)$ of the upper group $(Y > y_h) : I_h(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}{\overset{+}{M}_{h.}(Y)}$. The synthetic

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index $I(Y)$ is the arithmetic mean $M(\cdot)$ of the point measure: $I(Y) = M(I_h(Y))$. The decompositions of $I(Y)$ by subpopulations (Radaelli 2008, 2010; Zenga, 2016) and by sources (Zenga, Radaelli and Zenga 2012) have been obtained by the use of a two-step approach. In particular: *additive* decompositions of $I_h(Y)$ are obtained in the first step, and putting these decompositions in $M(I_h(Y))$, the corresponding decompositions of $I(Y)$ are easily derived.

Recently, Zenga (2013) has used this two-step approach for the decomposition by sources of the Gini (1914) and the Bonferroni (1930) indexes too. Moreover, by the use of the two-step approach the decomposition by subpopulations and by sources of the ζ Zenga (1984) index has been obtained by Porro and Zenga (2014) and by Arcagni et al (2014) respectively.

In the present paper, we use a two-step approach to obtain the ‘‘joint'' decomposition by subpopulations and sources of the inequality index $I(Y)$.

The paper is organized as follows. In the next section, some definitions and notation are introduced. In particular this section provides the definitions: of the lower and upper means in the whole population, and of the point $I_h(Y)$ and synthetic $I(Y)$ indexes. Section 3 illustrates the two-step approach for the additive decomposition by sources of $I_h(Y)$ and $I(Y)$. The bivariate distribution of the N units, according to the k different subpopulations and the r distinct values of the total income Y plays a significant role in the decomposition by subpopulations illustrated in Section 4. This section obtains an additive decomposition of $I_h(Y)$ in $k \times k$ non-negative contributions $B_{hlg}(Y)$. In Section 5, first of all, the contribution $B_{hlg}(Y)$ is additively decomposed into the c contributions $B_{hlg}(X_j)$ of each component (source) of the sum Y. In this way, the *basic* $k \times k \times c$ joint decomposition by subpopulations and sources of $I_h(Y)$ is achieved: $I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{hlg}(X_j)$. Moreover, Subsection 5.2.1 derives, through two different aggregations of the contributions $B_{hlg}(X_j)$, the decomposition by sources, and the decomposition by subpopulations. Section 6 provides the decomposition by sources of the point $I_{hl}(Y)$ and the synthetic $I_{.l}(Y)$ inequality indexes of the subpopulations. Section 7 provides an application to the net disposable income Y of the Italian households partitioned into three macro-regions. The total income Y is the sum of four components. The data used in this application are supplied by the 2012 Central Bank of Italy sample survey on household income and wealth (Bank of Italy, 2014). This survey covers $N = 8151$ households. In detail: Subsection 7.1 reports some aggregate characteristics in the three macro-regions; Subsection 7.2 provides the joint decomposition of three point measures $I_h(Y)$ and of the synthetic index $I(Y)$; in Subsection 7.2.2 particular attention is devoted to the *within* part of $I_h(Y)$ and of $I(Y)$; Subsection 7.2.3 gives the relative contributions of the four sources to the (three) point indexes and to the synthetic index $I(Y)$, as well as their relative shares to the subpopulation contribution $B_{hl.}(Y)$. In addition, Subsection 7.3 compares the decomposition by sources of each Italian macro-region with the joint decomposition by subpopulations and sources. Finally, Section 8 is devoted to the principal results and conclusions.

2. DEFINITIONS AND NOTATION

Let, $X_1, \ldots, X_j, \ldots, X_c$ be non-negative variates (income sources) observable on each of the N units of the population, and $Y = \sum_{j=1}^{c} X_j$ be the total income. Let: $\{0 \leq y_1 < \ldots < y_h < \ldots < y_r\}$ be the set of the distinct values assumed by the variate Y and $\{n_{1.}, \ldots, n_{h.}, \ldots, n_{r.}\}$ be the corresponding frequencies: $\sum_{h=1}^{r} n_{h.} = N$.

At each y_h the whole population can split into two non overlapping groups:

- a lower group with $\{Y \leq y_h\}$ including the first

$$P_{h.} = \sum_{t=1}^{h} n_{t.} \text{ units with cumulative income } Q_{h.}(Y) = \sum_{t=1}^{h} y_t \cdot n_{t.}, \tag{1}$$

and the corresponding

- upper group with $\{Y > y_h\}$ including the remaining $N - P_{h.}$ units, with income $T(Y) - Q_{h.}(Y)$, where $T(Y) = Q_{r.}(Y)$.

Let

$$\bar{M}_{h.}(Y) = \frac{Q_{h.}(Y)}{P_{h.}}, \qquad h = 1, \ldots, r, \tag{2}$$

be the arithmetic mean (lower mean) of the lower group, and

$$\overset{+}{M}_{h.}(Y) = \begin{cases} \frac{T(Y) - Q_{h.}(Y)}{N - P_{h.}}, & h = 1, \ldots, r-1 \\ \overset{+}{M}_{r-1.}(Y) = y_r, & h = r, \end{cases} \tag{3}$$

be the arithmetic mean (upper mean) of the upper group.

The Zenga (2007) point $I_h(Y)$ and synthetic $I(Y)$ indexes are given by:

$$I_h(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}{\overset{+}{M}_{h.}(Y)}, \quad h = 1, \ldots, r, \tag{4}$$

$$I(Y) = \sum_{h=1}^{r} I_h(Y) \cdot \frac{n_{h.}}{N} \; . \tag{5}$$

The following two popular properties of the arithmetic mean M play an important role in the decomposition by sources, by subpopulations, and in the joint decomposition by subpopulations and sources.

Property I. *Let, $X_1, \ldots, X_j, \ldots, X_c$ be c variates observable on each of the N units of a population and $Y = \sum_{j=1}^{c} X_j$ be their sum. Then:*

$$M(Y) = \sum_{j=1}^{c} M(X_j).$$

Property II. *The N units of a population are split in k different subpopulations with their corresponding frequencies $\{n_{.1}, \ldots n_{.g}, \ldots, n_{.k} : \sum_{g=1}^{k} n_{.g} = N\}$. Then:*

$$M = \sum_{g=1}^{k} M_g \cdot \frac{n_{.g}}{N},$$

where M is the mean of the whole population and M_g is the mean of subpopulation g.

3. DECOMPOSITION BY INCOME SOURCES

Let, $Q_{h.}(X_j)$, $j = 1, \ldots, c, h = 1, \ldots, r$ be the sum of the values of X_j observable in the lower group: $Q_{r.}(X_j) = T(X_j)$ denotes the sum of all the N values of X_j.

Let,

$$\bar{M}_{h.}(X_j) = \frac{Q_{h.}(X_j)}{P_{h.}}, \qquad h = 1, \ldots, r \tag{6}$$

be the mean of X_j in the lower group, and

$$\overset{+}{M}_{h.}(X_j) = \begin{cases} \dfrac{T(X_j) - Q_{h.}(X_j)}{N - P_{h.}}, & h = 1, \ldots, r-1 \\ \overset{+}{M}_{r-1.}(X_j) & h = r \end{cases}, \tag{7}$$

be the mean of X_j in the upper group. Then, from Property I:

$$\bar{M}_{h.}(Y) = \sum_{j=1}^{c} \bar{M}_{h.}(X_j), \text{ and } \overset{+}{M}_{h.}(Y) = \sum_{j=1}^{c} \overset{+}{M}_{h.}(X_j). \tag{8}$$

Now, by the use of these relations (8) in the numerator of (4), we have:

$$I_h(Y) = \frac{\sum_{j=1}^{c} \overset{+}{M}_{h.}(X_j) - \sum_{j=1}^{c} \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y)} = \sum_{j=1}^{c} B_h(X_j), \tag{9}$$

where

$$B_h(X_j) = \frac{1}{\overset{+}{M}_{h.}(Y)} \cdot \left(\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)\right) \tag{10}$$

is the contribution of X_j to $I_h(Y)$.

Finally, putting the decomposition (9) in (5), the decomposition by sources of $I(Y)$ is obtained:

$$I(Y) = \sum_{h=1}^{r} \left\{ \sum_{j=1}^{c} B_h(X_j) \right\} \cdot \frac{n_{h.}}{N} = \sum_{j=1}^{c} \sum_{h=1}^{r} B_h(X_j) \cdot \frac{n_{h.}}{N}$$

$$= \sum_{j=1}^{c} B(X_j), \tag{11}$$

where

$$B(X_j) = \sum_{h=1}^{r} B_h(X_j) \cdot \frac{n_{h.}}{N} \tag{12}$$

is the contribution of X_j to $I(Y)$, and it is equal to the arithmetic mean of the contributions of X_j to the point indexes $I_h(Y)$.

TABLE 1. - *Bivariate distribution of the N units according to the r values of Y and the k subpopulations*

	Subpopulation					
	1	…	g	…	k	Total
y_1	n_{11}	…	n_{1g}	…	n_{1k}	$n_{1.}$
⋮	⋮	⋱	⋮	⋱	⋮	⋮
y_h	n_{h1}	…	n_{hg}	…	n_{hk}	$n_{h.}$
⋮	⋮	⋱	⋮	⋱	⋮	⋮
y_r	n_{r1}	…	n_{rg}	…	n_{rk}	$n_{r.}$
Total	$n_{.1}$	…	$n_{.g}$	…	$n_{.k}$	N

4. DECOMPOSITIONS BY SUBPOPULATIONS

In the case of the decomposition by subpopulations we need the bivariate distribution of the N units according to k different subpopulations and the r distinct values of Y . This distribution is reported in Table 1, where n_{hg} denotes the frequency of the value y_h in the subpopulation g and $n_{.g}$ is the size of the subpopulation g.

For the distribution $\{(y_h, n_{hg}) : h = 1, \ldots, r\}$ of the subpopulation g the analogous of (1) are:

$$P_{hg} = \sum_{t=1}^{h} n_{tg} \text{ and } Q_{hg}(Y) = \sum_{t=1}^{h} y_t \cdot n_{tg}. \tag{13}$$

Moreover:

$$T_g(Y) = Q_{rg}(Y) \text{ and } M_g(Y) = T_g(Y)/n_{.g}.$$

In addition, for the subpopulation g, let

$$\begin{cases} y_{o(g)}, \text{ where } o(g) & = \min h : \; n_{hg} > 0 \\ y_{u(g)}, \text{ where } u(g) & = \max h : \; n_{hg} > 0, \end{cases} \tag{14}$$

and define the lower mean $\bar{M}_{hg}(Y)$ and the upper mean $\overset{+}{M}_{hg}(Y)$ as follows:

$$\bar{M}_{hg}(Y) = \begin{cases} y_{o(g)} & \text{for } h < o(g) \\ Q_{hg}(Y)/P_{hg} & \text{for h} \geq \text{o(g);} \end{cases} \tag{15}$$

$$\overset{+}{M}_{hg}(Y) = \begin{cases} \frac{T_g(Y) - Q_{hg}(Y)}{n_{.g} - P_{hg}} & \text{for h<u(g)} \\ y_{u(g)} & \text{for } h \geq u(g). \end{cases} \tag{16}$$

The decomposition by subpopulations of $I_h(Y)$ is obtained starting from the decomposition of the difference $[\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)]$. This decomposition is achieved by the use of the property II. In particular:

$$\bar{M}_{h.}(Y) = \sum_{l=1}^{k} \bar{M}_{hl}(Y) \cdot p(l|h) \quad \text{and} \tag{17}$$

$$\overset{+}{M}_{h.}(Y) = \sum_{l=1}^{k} \overset{+}{M}_{hl}(Y) \cdot a(l|h); \quad \text{where} \tag{18}$$

$p(l|h) = P_{hl}/P_{h.}$, $h = 1, \ldots, r$ is the relative frequency of subpopulation l in the lower group and

$$a(l|h) = \begin{cases} (n_{.l} - P_{hl})/(N - P_{h.}), & h = 1, \ldots, r-1 \\ n_{rl}/n_{r.}, & h = r \end{cases}$$

is the relative frequency of subpopulation l in the upper group.

With simple algebra, using the above reported expressions for $\overset{+}{M}_{h.}(Y)$ and $\bar{M}_{h.}(Y)$, Zenga (2016, sec.3) has obtained the following *basic* $k \times k$ additive decomposition of $I_h(Y)$:

$$\begin{aligned} I_h(Y) &= \sum_{l=1}^{k} \sum_{g=1}^{k} \left[\frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(g|h) \\ &= \sum_{l=1}^{k} \sum_{g=1}^{k} B_{hlg}(Y) = B_{h..}(Y). \end{aligned} \tag{19}$$

It is worth to remark that, starting from the $k \times k$ decomposition (19), it is possible to obtain many others additive decompositions of $I_h(Y)$.

Thus, after some steps we obtain:

$$\sum_{g=1}^{k} B_{hlg}(Y) = B_{hl.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l|h), \tag{20}$$

and by the use of (20) in (19) we have

$$I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y) = \sum_{l=1}^{k} \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l|h). \tag{21}$$

Formula (21) shows that, the point index $I_h(Y)$ is the weighted mean of the k relative variations $\left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right]$ with weights $p(l|h)$. Thus, $B_{hl.}(Y)$ can be interpreted as the contribution of the subpopulation l to the point inequality index $I_h(Y)$.

Now, $B_{hl.}(Y)$ can be split into a within and a between components.

$$B_{hl.}(Y) = \sum_{g=1}^{k} B_{hlg}(Y) = B_{hll}(Y) + \sum_{(g:g \neq l)} B_{hlg}(Y),$$

$$B_{hl.}(Y) = B_{hlW}(Y) + B_{hlB}(Y). \tag{22}$$

Consequently, the within and the between component of the point index $I_h(Y)$ are given by.

$$\begin{aligned} I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y) &= \sum_{l=1}^{k} B_{hlW}(Y) + \sum_{l=1}^{k} B_{hlB}(Y) \\ &= B_{h.W}(Y) + B_{h.B}(Y). \end{aligned} \tag{23}$$

In (23) $B_{h.W}(Y) = \sum_{l=1}^{k} B_{hlW}(Y)$ can be interpreted as the within contribution of all the subpopulations to $I_h(Y)$, and $B_{h.B}(Y) = \sum_{l=1}^{k} B_{hlB}(Y)$ can be interpreted as the between contribution of all the subpopulations to $I_h(Y)$.

Finally, putting the decomposition (19) of $I_h(Y)$ in (5), the following decomposition of the synthetic index is obtained:

$$I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \left\{ \sum_{h=1}^{r} B_{hlg}(Y) \cdot \frac{n_{h.}}{N} \right\} = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{.lg}(Y) = B_{...}(Y). \tag{24}$$

In (24) $B_{.lg}(Y)$ is the weighted mean of the r contributions $B_{hlg}(Y)$. Now, from (24) we obtain:

$$I(Y) = \sum_{l=1}^{k} \left\{ B_{.ll}(Y) + \sum_{(g:g \neq l)} B_{.lg}(Y) \right\} = \sum_{l=1}^{k} B_{.l.}(Y) \tag{25}$$

where

$$B_{.l.}(Y) = \sum_{g=1}^{k} B_{.lg}(Y) \tag{26}$$

is the contribution of subpopulation l to $I(Y)$.

Moreover, from the relation (25) we have

$$I(Y) = \sum_{l=1}^{k} \{ B_{.lW}(Y) + B_{.lB}(Y) \}, \tag{27}$$

and

$$I(Y) = B_{..W}(Y) + B_{..B}(Y). \tag{28}$$

In (27) $B_{.lW} = \sum_{h=1}^{r} B_{hll}(Y) \cdot \frac{n_{h.}}{N}$ and $B_{.lB}(Y) = \sum_{(g:g \neq l)} \left\{ \sum_{h=1}^{r} B_{hlg}(Y) \cdot \frac{n_{h.}}{N} \right\}$ are the within and between parts of $B_{.l.}(Y)$, respectively. In (28)

$$B_{..W}(Y) = \sum_{l=1}^{k} \left\{ \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(l|h) \cdot \frac{n_{h.}}{N} \right\}, \tag{29}$$

and

$$B_{..B}(Y) = \sum_{l=1}^{k} \sum_{(g:g\neq l)} \left\{ \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(g|h) \cdot \frac{n_{h.}}{N} \right\} \tag{30}$$

are the within and the between parts of the synthetic index $I(Y)$, respectively. Note that in $B_{..W}(Y)$ only comparisons between upper and lower means of the same subpopulations are involved, while in $B_{..B}(Y)$ only comparisons between upper and lower means of different subpopulations are involved.

5. JOINT DECOMPOSITION OF $I_h(Y)$ AND $I(Y)$ BY SUBPOPULATIONS AND SOURCES

To get the joint decomposition we need, for each subpopulation g, the lower means $\bar{M}_{hg}(X_j)$ and the upper means $\overset{+}{M}_{hg}(X_j)$ of the c sources. Subsection 5.1 introduces the lower means $\bar{M}_{hg}(X_j)$ and the upper means $\overset{+}{M}_{hg}(X_j)$ of the c sources and their relationship with the corresponding lower and upper means of the sum Y. Then, in Subsection 5.2 the joint decomposition by subpopulations and sources is achieved. Subsection 5.2.1 derives from the joint decomposition the corresponding decompositions by subpopulations and by sources. Finally, the decompositions by sources for each subpopulation is illustrated in Subsection 5.3.

5.1 *Lower and Upper Means of the Sources for Subpopulations*

Let,

$$S_{hg}(Y) = y_h \cdot n_{hg}, h = 1, \ldots, r. \tag{31}$$

Thus, from (13),

$$Q_{hg}(Y) = \sum_{t=1}^{h} y_t \cdot n_{tg} = \sum_{t=1}^{h} S_{tg}(Y). \tag{32}$$

In the case $n_{hg} > 0$, let us denote with $x_{hgjd}, \quad d = 1, \ldots, n_{hg}$, the values of X_j observable on each of the n_{hg} units of the subpopulation g with total income $Y = y_h$. Obviously, in this case, $\sum_{j=1}^{c} x_{hgjd} = y_h, \forall d = 1, \ldots, n_{hg}, \forall h = 1, \ldots, r, \forall g = 1, \ldots, k$.

Let:

$$S_{hg}(X_j) = \begin{cases} 0, & \text{for } n_{hg} = 0 \\ \sum_{d=1}^{n_{hg}} x_{hgjd}, & \text{for } n_{hg} > 0; \end{cases} \tag{33}$$

$$M_{hg}(X_j) = \frac{S_{hg}(X_j)}{n_{hg}}, \quad \text{for } n_{hg} > 0, \tag{34}$$

$$M_{hg}(Y) = \frac{S_{hg}(Y)}{n_{hg}} = y_h, \text{ for } n_{hg} > 0.$$

Note that:

$$\sum_{d=1}^{n_{hg}}\sum_{j=1}^{c} x_{hgjd} = \sum_{d=1}^{n_{hg}} y_h = y_h \cdot n_{hg} = S_{hg}(Y),$$

$$\sum_{j=1}^{c}\sum_{d=1}^{n_{hg}} x_{hgjd} = \sum_{j=1}^{c} S_{hg}(X_j) = S_{hg}(Y), \tag{36}$$

$$\sum_{j=1}^{c} M_{hg}(X_j) \cdot n_{hg} = M_{hg}(Y) \cdot n_{hg} \tag{37}$$

Let:

$$Q_{hg}(X_j) = \sum_{t=1}^{h} S_{tg}(X_j), \tag{38}$$

be the sum of the values of X_j observable on each of the P_{hg} units of the subpopulation g with $Y \leq y_h$. Moreover:

$$T_g(X_j) = Q_{rg}(X_j) = \sum_{h=1}^{r} S_{hg}(X_j),$$

be the sum of the $n_{.g}$ values of X_j of the subpopulation g, and

$$M_g(X_j) = \frac{T_g(X_j)}{n_{.g}}.$$

Now, from (38), (36) and (32):

$$\sum_{j=1}^{c} Q_{hg}(X_j) = \sum_{j=1}^{c}\sum_{t=1}^{h} S_{tg}(X_j) = \sum_{t=1}^{h}\sum_{j=1}^{c} S_{tg}(X_j) = \sum_{t=1}^{h} S_{tg}(Y)$$
$$= Q_{hg}(Y). \tag{39}$$

In coherence with (15) and (16) let:

$$\bar{M}_{hg}(X_j) = \begin{cases} M_{o(g)g}(X_j), & \text{for } h < o(g) \\ Q_{hg}(X_j)/P_{hg}, & \text{for } h \geq o(g); \end{cases} \tag{40}$$

$$\overset{+}{M}_{hg}(X_j) = \begin{cases} \frac{T_g(X_j) - Q_{hg}(X_j)}{n_{.g} - P_{hg}}, & \text{for } h < u(g) \\ M_{u(g)g}(X_j), & \text{for } h \geq u(g). \end{cases} \tag{41}$$

Finally,

$$\sum_{j=1}^{c} \bar{M}_{hg}(X_j) = \begin{cases} \sum_{j=1}^{c} & M_{o(g)g}(X_j) = M_{o(g)g}(Y), \text{ for h<o(g)} \\ \sum_{j=1}^{c} & \frac{Q_{hg}(X_j)}{P_{hg}} = \frac{Q_{hg}(Y)}{P_{hg}} = \bar{M}_{hg}(Y), \text{for h}\geq\text{o(g)} \end{cases}$$
$$= \bar{M}_{hg}(Y),\ h = 1, \ldots, r, \tag{42}$$

and

$$\sum_{j=1}^{c} \overset{+}{M}_{hg}(X_j) = \overset{+}{M}_{hg}(Y), h = 1, \ldots, r. \tag{43}$$

5.2 *From the Decomposition by Subpopulations to the Joint Decomposition*

By the use of the relations (42) and (43) in the decomposition by subpopulations (19) of $I_h(Y)$, the *basic* $k \times k \times c$ joint decomposition by subpopulations and sources of $I_h(Y)$ is obtained:

$$I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{hlg}(X_j), \tag{44}$$

where

$$B_{hlg}(X_j) = \left[\frac{\overset{+}{M}_{hg}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(g|h) \tag{45}$$

is the contribution to $I_h(Y)$ that derives from the comparison of the lower mean $\bar{M}_{hl}(X_j)$ w.r.t the upper mean $\overset{+}{M}_{hg}(X_j)$. Note that:

$$\sum_{j=1}^{c} B_{hlg}(X_j) = B_{hlg}(Y). \tag{46}$$

In other words the contribution $B_{hlg}(Y)$ is decomposed additively by the contributions $B_{hlg}(X_j)$ of each component X_j of the sum Y.

Now, after some steps we obtain:

$$B_{hl.}(Y) = \sum_{j=1}^{c} B_{hl.}(X_j), \tag{47}$$

where

$$B_{hl.}(X_j) = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)} \cdot p(l|h), \tag{48}$$

is the ‘‘additive’’ contribution of X_j to $B_{hl.}(Y)$. Moreover,

$$B_{hl.}(X_j) = B_{hlW}(X_j) + B_{hlB}(X_j). \tag{49}$$

where

$$B_{hlW}(X_j) = B_{hll}(X_j), \tag{50}$$

and

$$B_{hlB}(X_j) = \sum_{(g:g \neq l)} B_{hlg}(X_j), \tag{51}$$

are the within and the between parts of $B_{hl.}(X_j)$.

Now from (47) and (49) we have:

$$B_{hl.}(Y) = \sum_{j=1}^{c} \{B_{hlW}(X_j) + B_{hlB}(X_j)\}. \tag{52}$$

Finally, putting (52) in (23) the following "joint decompositions" of the point index $I_h(Y)$ are obtained:

$$I_h(Y) = \sum_{l=1}^{k} \sum_{j=1}^{c} \{B_{hlW}(X_j) + B_{hlB}(X_j)\};$$

$$I_h(Y) = \sum_{j=1}^{c} \sum_{l=1}^{k} \{B_{hlW}(X_j) + B_{hlB}(X_j)\}; \tag{53}$$

$$I_h(Y) = \sum_{j=1}^{c} \{B_{h.W}(X_j) + B_{h.B}(X_j)\}, \tag{54}$$

where

$$B_{h.W}(X_j) = \sum_{l=1}^{k} B_{hlW}(X_j) \tag{55}$$

and

$$B_{h.B}(X_j) = \sum_{l=1}^{k} B_{hlB}(X_j) \tag{56}$$

can be interpreted as the within and the between contributions of X_j to $I_h(Y)$.

Finally, putting the *basic* joint decomposition (44) of $I_h(Y)$ in (5), the $k \times k \times c$ joint decomposition (by subpopulations and sources) of $I(Y)$ is obtained:

$$I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} \left\{ \sum_{h=1}^{r} B_{hlg}(X_j) \frac{n_{h.}}{N} \right\},$$

$$I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{.lg}(X_j), \tag{57}$$

where

$$B_{.lg}(X_j) = \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hg}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l \mid h) \cdot a(g|h) \cdot \frac{n_{h.}}{N}, \tag{58}$$

is the weighted arithmetic mean of $B_{hlg}(X_j)$ with weights $\frac{n_{h.}}{N}$.

Starting from the $k \times k \times c$ joint decomposition (57) of $I(Y)$, it is possible to derive, with simple algebra many other decompositions. In Section 7 some of these decompositions will be illustrated.

5.2.1 *From the Joint Decomposition to the Decompositions by Subpopulations, and by sources*

Relation (46) shows that the sum $\sum_{j=1}^{c} B_{hlg}(X_j)$ of the c joint contributions $B_{hlg}(X_j)$ is equal to the contribution $B_{hlg}(Y)$ by subpopulations of the total income Y. We show now the following

Lemma 1
The sum $B_{h..}(X_j)$ of the $k \times k$ contributions $B_{hlg}(X_j)$,

$$B_{h..}(X_j) = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{hlg}(X_j) \tag{59}$$

is equal to contribution $B_h(X_j) = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y)}$ introduced in Section 3.
By the use of (45) *in* (59) *we have:*

$$B_{h..}(X_j) = \sum_{l=1}^{k} \sum_{g=1}^{k} \frac{\overset{+}{M}_{hg}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)} \cdot p(l \mid h) \cdot a(g|h) =$$

$$= \sum_{l=1}^{k} \frac{p(l \mid h)}{\overset{+}{M}_{h.}(Y)} \sum_{g=1}^{k} \overset{+}{M}_{hg}(X_j) \cdot a(g|h) - \sum_{g=1}^{k} \frac{a(g|h)}{\overset{+}{M}_{h.}(Y)} \sum_{l=1}^{k} \bar{M}_{hl}(X_j) \cdot p(l \mid h).$$

Finally, by the use of Property II the result is reached.

The contribution of X_j to $I(Y)$ is given by:

$$B_{...}(X_j) = \sum_{h=1}^{r} B_{h..}(X_j) \cdot \frac{n_{h.}}{N}, \tag{60}$$

and it is equal to the contribution $B(X_j)$ introduced in Section 3.

If $I_h(Y) > 0$, we may define the relative contribution of X_j to the point index $I_h(Y)$ by:

$$\beta_{h..}(X_j) = \frac{B_{h..}(X_j)}{I_h(Y)} = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}, \tag{61}$$

obviously $\sum_{j=1}^{c} \beta_{h..}(X_j) = 1$.

The relative contribution of X_j to $I(Y)$ is given by:

$$\beta_{...}(X_j) = \frac{B_{...}(X_j)}{I(Y)} = \sum_{h=1}^{r} \beta_{h..}(X_j) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N}. \tag{62}$$

It has been observed in Zenga *et al.* (2012) that, the relative contributions $\beta_{h..}(X_j)$ and $\beta_{...}(X_j)$ should be compared with the shares $\gamma_{..}(X_j) = \frac{M(X_j)}{M(Y)}$ in order to explain how income sources X_j affect total income inequality. If $\beta_{...}(X_j)$ is less (greater) than $\gamma_{..}(X_j)$, income sources X_j reduces (augments) inequality of total income distribution. More details on this point can be found in Zenga (2013), too.

6. DECOMPOSITION BY SOURCES OF THE POINT $I_{hl}(Y)$ AND THE SYNTHETIC $I_{.l}(Y)$ INEQUALITY INDEXES

The synthetic inequality index of the subpopulation l is given by:

$$I_{.l}(Y) = \sum_{h=1}^{r} I_{hl}(Y) \cdot \frac{n_{hl}}{n_{.l}} = \sum_{h=1}^{r} \frac{\overset{+}{M}_{hl}(Y) - \overset{-}{M}_{hl}(Y)}{\overset{+}{M}_{hl}(Y)} \cdot \frac{n_{hl}}{n_{.l}}, \tag{63}$$

where

$$I_{hl}(Y) = \frac{\overset{+}{M}_{hl}(Y) - \overset{-}{M}_{hl}(Y)}{\overset{+}{M}_{hl}(Y)} \tag{64}$$

is the point inequality index of the subpopulation l.

It can be useful to decompose by sources the point $I_{hl}(Y)$ and the synthetic $I_{.l}(Y)$ inequality indexes. By the use of property I in the numerator of (64) we have:

$$I_{hl}(Y) = \sum_{j=1}^{c} \frac{\overset{+}{M}_{hl}(X_j) - \overset{-}{M}_{hl}(X_j)}{\overset{+}{M}_{hl}(Y)} = \sum_{j=1}^{c} B_{hl}(X_j), \tag{65}$$

where

$$B_{hl}(X_j) = \left(\overset{+}{M}_{hl}(X_j) - \overset{-}{M}_{hl}(X_j)\right) / \overset{+}{M}_{hl}(Y) \tag{66}$$

is the contribution of X_j to $I_{hl}(Y)$.

Finally, putting the decomposition (65) in (63), the decomposition by sources of $I_{.l}(Y)$ is obtained:

$$I_{.l}(Y) = \sum_{h=1}^{r} \left\{ \sum_{j=1}^{c} B_{hl}(X_j) \right\} \cdot \frac{n_{hl}}{n_{.l}} = \sum_{j=1}^{c} B_{.l}(X_j) \tag{67}$$

where

$$B_{.l}(X_j) = \sum_{h=1}^{r} B_{hl}(X_j) \cdot \frac{n_{hl}}{n_{.l}}. \tag{68}$$

7. APPLICATION

The data used in this application are supplied by the 2012 Central Bank of Italy sample survey on household income and wealth (Bank of Italy 2014). This survey covers $N = 8151$ households.

In this paper we deal with the household net disposable income Y, that is the sum of: the payroll income X_1, the pensions and net transfers X_2, the net self employment income X_3, and the property incomes X_4. The $N = 8151$ households have been partitioned according to their residence area: North (1), Center (2) and South with Islands (3). In all computations that follow we consider the weights $w_i > 0$ $(i = 1, 2, \ldots, 8151;$ $W = \sum w_i = 8151 = N)$ supplied by the Central Bank of Italy for each household; these weights are defined as the inverse of household's probability of inclusion in the sample (For further details see Banca d'Italia 2014). Now we remark that, in the following sections we will not use the notation related to the weights w_i, but for simplicity's sake we will continue the use of the notation of the previous sections. Thus, to denote the sum of the weights of the n_{hl} households of the subpopulation l with total income $Y = y_h$ we will use n_{hl} instead of $w_{hl}, \ldots$. Note that the frequency distribution of the total income Y has $r = 7287$ different values.

7.1 *Aggregate Characteristic in Three Italian Macro-region*

Table 2 reports for each geographic area: the arithmetic means $M_l(X_j)$ and $M_l(Y)$ of the components X_j and of the total income Y; the shares $\gamma_{.l}(X_j) = M_l(X_j)/M_l(Y)$; the shares $\gamma_{..}(X_j) = M(X_j)/M(Y)$; the synthetic index $I_{.l}(Y)$; the sum of the weights $n_{.l}$; the relative weights $n_{.l}/N$; the shares $T_l(Y)/T(Y)$.

TABLE 2. - *Some aggregate charecteristics for geographic areas*

	North $l = 1$	Center $l = 2$	South $l = 3$	Italy
$n_{.l}$	3971.95	1537.37	2641.68	8151
$n_{.l}/N$	0.48729	0.18861	0.32409	1.0000
$I_{.l}(Y)$	0.6949	0.6592	0.6919	$0.70014 = I(Y)$
Means of Y and X_j, and shares $T_l(Y)/T(Y)$				
X_1	13595.47	13032.01	9396.36	12128.29
X_2	8965.00	9226.36	6951.88	8361.86
X_3	3688.90	3683.57	2559.80	3321.96
X_4	7293.81	8058.15	4609.82	6568.11
Y	33543.17	34000.09	23517.86	30380.22
$T_l(Y)/T(Y)$	0.5380	0.2111	0.2509	1.0000
Shares: $\gamma_{.l}(X_j)$ and $\gamma_{..}(X_j)$				
$\gamma_{.l}(X_1)$	0.4053	0.3833	0.3995	$0.3992 = \gamma_{..}(X_1)$
$\gamma_{.l}(X_2)$	0.2673	0.2714	0.2956	$0.2752 = \gamma_{..}(X_2)$
$\gamma_{.l}(X_3)$	0.1100	0.1083	0.1088	$0.1093 = \gamma_{..}(X_3)$
$\gamma_{.l}(X_4)$	0.2174	0.2370	0.1960	$0.2162 = \gamma_{..}(X_4)$
	1.0000	1.0000	1.0000	1.0000

Table 2 shows that the mean value of the South is very far from the means of the other two Italian macro-regions, while the shares $\gamma_{.l}(X_j)$ of the three macro regions are very similar and that their differences with the corresponding shares $\gamma_{..}(X_j)$ of the whole country are negligible. Note that: $\gamma_{..}(X_j) = \sum_{l=1}^{k} \gamma_{.l}(X_j) \cdot T_l(Y)/T(Y)$. Moreover, Table 2 shows that the North has the greatest inequality, while the Center has the lowest one, and the inequality of the whole population is a little bit greater than the one of the North. The synthetic inequality index $I(Y)$ =0.70014 means that in the whole population, on average, the lower mean is equal to the (1-0.70014)·100 $\simeq$ 30% of the upper mean.

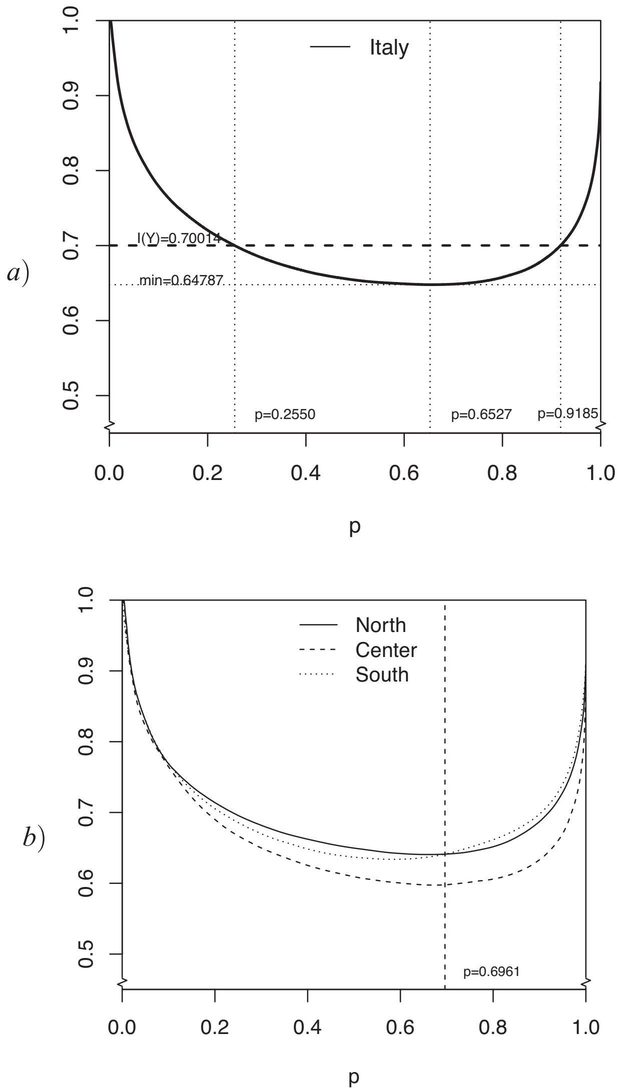

FIGURE 1. - *Graphs of the point measure for geographic areas*

Figure 1 displays the graphs of the point inequality measures for: *a*) the whole population; *b*) the North, the Center and the South. For the subpopulation l the abscissas and the ordinates are given respectively by

$$p_{hl} = \frac{P_{hl}}{n_{.l}} \text{ and } I_{(p_{hl})l}(Y) = I_{hl}(Y), \forall h = 1, \ldots, r,$$

while for the whole population the abscissas and the ordinates are given respectively by

$$p_{h.} = \frac{P_{h.}}{N} \text{ and } I_{(p_{h.})}(Y) = I_h(Y), \forall h = 1, \ldots, r$$

7.2 *Joint Decomposition by Geographical Areas and Sources of the Point and the Synthetic Inequality Indexes*

The decomposition of $I_h(Y)$ and $I(Y)$ by subpopulations is widely illustrated in Zenga (2016), while the case of the decomposition by sources is presented in Zenga *et al.* (2012). In this section we illustrate the decompositions of the point measure $I_{h(p)}(Y) = I_{(p)}(Y)$ for three values of p, and the decompositions of the synthetic index $I(Y) = 0.70014$. For p we have chosen the following values:

- $p = 0.10$, because $I_{(0.10)}(Y) = 0.7793$ compares the income mean of the poorest 10% households with the income mean of the other 90% households;
- $p = 0.50$, because $I_{(0.50)}(Y) = 0.654$ compares the income mean of the households with $Y \leq Median(Y)$ with the mean income of the households with $Y > Median(Y)$;
- $p = 0.95$, because $I_{(0.95)}(Y) = 0.7282$ compares the income mean of the lower group that is the 95% of the whole population with the income mean of the reachest 5% of the households.

Tables 3, 4 and 5 report the joint decompositions of $I_{(0.10)}(Y)$=0.7793, of $I_{(0.50)}(Y)$=0.6540 and of $I_{(0.95)}(Y) = 0.7282$, as well as all the values necessary for their computations. Moreover, Table 6 reports the joint decompositions of the synthetic index $I(Y) = 0.7001$; it is useful to remember that the contributions reported in this latter table are the weighted arithmetic means of the corresponding contributions of the r point indexes $I_h(Y)$ with weights $n_{h.}/N$. For example, the value $B_{.3.}(Y) = 0.3100$, reported in Table 6, is the weighted mean of the $r = 7287$ contributions $B_{h3.}(Y) = \frac{\overset{+}{M}_{h.}(Y) - \overset{-}{M}_{h3}(Y)}{\overset{+}{M}_{h.}(Y)} \cdot p(3|h)$ with weights $n_{h.}/N$.

TABLE 3. - *Upper and lower means in the geographic areas and joint decomposition of $I_{(0.1)}(Y) = 0.7793$ by subpopulations and sources*

$p = 0.10; h = 460$	North $l = 1$	Center $l = 2$	South $l = 3$	Italy
$Y \leq 10600$	275.78	114.01	425.40	815.2
$Y > 10600$	3696.16	1423.36	2216.28	7335.80
Total $n_{.l}$	3971.95	1537.37	2641.68	8151

Conditional relative frequencies

$p(l\|h)$	0.3383	0.1399	0.5218	1.0000
$a(g\|h)$	0.50385	0.19403	0.3021	1.0000

Lower and Upper means

$\overset{-}{M}_{hl}(X_1)$	2790.99	2096.46	1464.93	2001.86
$\overset{+}{M}_{hl}(X_1)$	14401.62	13907.97	10918.75	13253.60
$\overset{-}{M}_{hl}(X_2)$	3144.73	3294.99	4001.58	3612.88
$\overset{+}{M}_{hl}(X_2)$	9399.27	9701.48	7518.17	8889.59
$\overset{-}{M}_{hl}(X_3)$	250.03	708.65	222.01	299.55
$\overset{+}{M}_{hl}(X_3)$	3945.49	3921.86	3008.53	3657.83
$\overset{-}{M}_{hl}(X_4)$	905.70	1453.95	1621.52	1355.92
$\overset{+}{M}_{hl}(X_4)$	7770.44	8587.16	5183.40	7147.32
$\overset{-}{M}_{hl}(Y)$	7091.45	7554.05	7310.03	7270.21
$\overset{+}{M}_{hl}(Y)$	35516.81	36118.46	26628.84	32948.33

Joint decomposition

$B_{(0.1)lW}(X_1)$	0.0601	0.0097	0.0452	$0.1150 = B_{(0.1).W}(X_1)$
$B_{(0.1)lB}(X_1)$	0.0474	0.0376	0.1415	$0.2265 = B_{(0.1).B}(X_1)$
$B_{(0.1)l.}(X_1)$	0.1074	0.0474	0.1867	$0.3415 = B_{(0.1)..}(X_1)$
$B_{(0.1)lW}(X_2)$	0.0324	0.0053	0.0168	$0.0545 = B_{(0.1).W}(X_2)$
$B_{(0.1)lB}(X_2)$	0.0266	0.0185	0.0606	$0.1057 = B_{(0.1).B}(X_2)$
$B_{(0.1)l.}(X_2)$	0.0590	0.0237	0.0774	$0.1602 = B_{(0.1)..}(X_2)$
$B_{(0.1)lW}(X_3)$	0.0191	0.0026	0.0133	$0.0351 = B_{(0.1).W}(X_3)$
$B_{(0.1)lB}(X_3)$	0.0159	0.0099	0.0411	$0.0668 = B_{(0.1).B}(X_3)$
$B_{(0.1)l.}(X_3)$	0.0350	0.0125	0.0544	$0.1019 = B_{(0.1)..}(X_3)$
$B_{(0.1)lW}(X_4)$	0.0355	0.0059	0.0170	$0.0584 = B_{(0.1).W}(X_4)$
$B_{(0.1)lB}(X_4)$	0.0286	0.0183	0.0705	$0.1173 = B_{(0.1).B}(X_4)$
$B_{(0.1)l.}(X_4)$	0.0640	0.0242	0.0875	$0.1757 = B_{(0.1)..}(X_4)$
$B_{(0.1)lW}(Y)$	0.1471	0.0235	0.0924	$0.2630 = B_{(0.1).W}(Y)$
$B_{(0.1)lB}(Y)$	0.1184	0.0843	0.3137	$0.5164 = B_{(0.1).B}(Y)$
$B_{(0.1)l.}(Y)$	0.2655	0.1078	0.4061	$I_{(0.1)}(Y) = 0.7793$

TABLE 4. - *Upper and lower means in the geographic areas and joint decomposition of $I_{(0.5)}(Y) = 0.6540$ by subpopulations and sources*

$p = 0.50; h = 3064$	North $l = 1$	Center $l = 2$	South $l = 3$	Italy
$Y \leq 24590.1$	1710.1	576.9	1788.6	4075.6
$Y > 24590.1$	2261.8	960.44	853.07	4075.4
Total $n_{.l}$	3971.95	1537.37	2641.68	8151

Conditional relative frequencies

$p(l\|h)$	0.4196	0.1415	0.4388	1.0000
$a(g\|h)$	0.5550	0.2357	0.2093	1.0000

Lower and Upper means

$\overset{-}{M}_{hl}(X_1)$	6022.17	5334.67	4783.94	5381.45
$\overset{+}{M}_{hl}(X_1)$	19321.40	17655.72	19067.15	18875.63
$\overset{-}{M}_{hl}(X_2)$	6325.13	6230.65	6180.62	6248.34
$\overset{+}{M}_{hl}(X_2)$	10960.92	11025.86	8568.96	10475.53
$\overset{-}{M}_{hl}(X_3)$	590.76	843.54	1029.69	819.17
$\overset{+}{M}_{hl}(X_3)$	6031.31	5389.54	5767.96	5824.94
$\overset{-}{M}_{hl}(X_4)$	3075.27	4041.06	2977.91	3169.25
$\overset{+}{M}_{hl}(X_4)$	10483.31	10471.18	8031.40	9967.21
$\overset{-}{M}_{hl}(Y)$	16013.33	16449.92	14972.15	15618.21
$\overset{+}{M}_{hl}(Y)$	46796.94	44542.29	41435.48	45143.31

Joint decomposition

$B_{(0.5)lW}(X_1)$	0.0686	0.0091	0.0291	$0.1068 = B_{(0.5).W}(X_1)$
$B_{(0.5)lB}(X_1)$	0.0509	0.0334	0.1079	$0.1921 = B_{(0.5).B}(X_1)$
$B_{(0.5)l.}(X_1)$	0.1195	0.0425	0.1370	$0.2989 = B_{(0.5)..}(X_1)$
$B_{(0.5)lW}(X_2)$	0.0239	0.0035	0.0049	$0.0323 = B_{(0.5).W}(X_2)$
$B_{(0.5)lB}(X_2)$	0.0147	0.0098	0.0369	$0.0613 = B_{(0.5).B}(X_2)$
$B_{(0.5)l.}(X_2)$	0.0386	0.0133	0.0418	$0.0936 = B_{(0.5)..}(X_2)$
$B_{(0.5)lW}(X_3)$	0.0281	0.0034	0.0096	$0.0411 = B_{(0.5).W}(X_3)$
$B_{(0.5)lB}(X_3)$	0.0206	0.0123	0.0370	$0.0698 = B_{(0.5).B}(X_3)$
$B_{(0.5)l.}(X_3)$	0.0486	0.0156	0.0466	$0.1109 = B_{(0.5)..}(X_3)$
$B_{(0.5)lW}(X_4)$	0.0382	0.0048	0.0103	$0.0533 = B_{(0.5).W}(X_4)$
$B_{(0.5)lB}(X_4)$	0.0258	0.0138	0.0577	$0.0973 = B_{(0.5).B}(X_4)$
$B_{(0.5)l.}(X_4)$	0.0641	0.0186	0.0679	$0.1506 = B_{(0.5)..}(X_4)$
$B_{(0.5)lW}(Y)$	0.1588	0.0208	0.0539	$0.2335 = B_{(0.5).W}(Y)$
$B_{(0.5)lB}(Y)$	0.1120	0.0692	0.2394	$0.4206 = B_{(0.5).B}(Y)$
$B_{(0.5)l.}(Y)$	0.2707	0.0900	0.2933	$0.6540 = I_{(0.5)}(Y)$

TABLE 5. - *Upper and lower means in the geographic areas and joint decomposition of $I_{(0.95)}(Y) = 0.7282$ by subpopulations and sources*

$p = 0.95$; $h = 6841$	North $l = 1$	Center $l = 2$	South $l = 3$	Italy
$Y \leq 68819.2$	3710.67	1443.33	2589.48	7743.48
$Y > 68819.2$	261.28	94.04	52.20	407.52
Total $n_{.l}$	3971.95	1537.37	2641.68	8151

Conditional relative frequencies

$p(l\|h)$	0.4792	0.1864	0.3344	1.0000
$a(g\|h)$	0.6411	0.2308	0.1281	1.0000

Lower and Upper means

$\overset{-}{M}_{hl}(X_1)$	12423.02	11591.13	8975.15	11114.97
$\overset{+}{M}_{hl}(X_1)$	30246.47	35146.50	30292.69	31383.15
$\overset{-}{M}_{hl}(X_2)$	8339.86	8704.60	6857.59	7912.16
$\overset{+}{M}_{hl}(X_2)$	17843.19	17234.43	11629.28	16906.81
$\overset{-}{M}_{hl}(X_3)$	2163.54	2925.30	1835.53	2195.83
$\overset{+}{M}_{hl}(X_3)$	25352.04	15321.32	38491.26	24720.22
$\overset{-}{M}_{hl}(X_4)$	5930.53	7180.42	4151.20	5568.48
$\overset{+}{M}_{hl}(X_4)$	26654.97	21529.53	27361.78	25562.73
$\overset{-}{M}_{hl}(Y)$	28856.94	30401.45	21819.47	26791.44
$\overset{+}{M}_{hl}(Y)$	100096.67	89231.78	107775.02	98572.91

Joint decomposition

$B_{(0.95)lW}(X_1)$	0.0556	0.0103	0.0093	$0.0751 = B_{(0.95).W}(X_1)$
$B_{(0.95)lB}(X_1)$	0.0366	0.0271	0.0668	$0.1305 = B_{(0.95).B}(X_1)$
$B_{(0.95)l.}(X_1)$	0.0922	0.0374	0.0760	$0.2056 = B_{(0.95)..}(X_1)$
$B_{(0.95)lW}(X_2)$	0.0296	0.0037	0.0021	$0.0354 = B_{(0.95).W}(X_2)$
$B_{(0.95)lB}(X_2)$	0.0120	0.0118	0.0320	$0.0558 = B_{(0.95).B}(X_2)$
$B_{(0.95)l.}(X_2)$	0.0416	0.0155	0.0341	$0.0912 = B_{(0.95)..}(X_2)$
$B_{(0.95)lW}(X_3)$	0.0723	0.0054	0.0159	$0.0936 = B_{(0.95).W}(X_3)$
$B_{(0.95)lB}(X_3)$	0.0374	0.0358	0.0617	$0.1349 = B_{(0.95).B}(X_3)$
$B_{(0.95)l.}(X_3)$	0.1097	0.0412	0.0776	$0.2285 = B_{(0.95)..}(X_3)$
$B_{(0.95)lW}(X_4)$	0.0646	0.0063	0.0101	$0.0809 = B_{(0.95).W}(X_4)$
$B_{(0.95)lB}(X_4)$	0.0308	0.0285	0.0626	$0.1219 = B_{(0.95).B}(X_4)$
$B_{(0.95)l.}(X_4)$	0.0954	0.0348	0.0726	$0.2028 = B_{(0.95)..}(X_4)$
$B_{(0.95)lW}(Y)$	0.2220	0.0257	0.0373	$0.2850 = B_{(0.95).W}(Y)$
$B_{(0.95)lB}(Y)$	0.1169	0.1032	0.2231	$0.4432 = B_{(0.95).B}(Y)$
$B_{(0.95)l.}(Y)$	0.3389	0.1289	0.2604	$0.7282 = I_{(0.95)}(Y)$

TABLE 6. - *Joint decomposition of* $I(Y) = B_{...}(Y) = 0.7001$. *Contributions of within and between components for three regions and four income sources*

	North $l = 1$	Center $l = 2$	South $l = 3$	Italy
$B_{.lW}(X_1)$	0.0661	0.0091	0.0288	$0.1040 = B_{..W}(X_1)$
$B_{.lB}(X_1)$	0.0478	0.0329	0.1063	$0.1870 = B_{..B}(X_1)$
$B_{.l.}(X_1)$	0.1140	0.0420	0.1350	$0.2910 = B_{...}(X_1)$
$B_{.lW}(X_2)$	0.0267	0.0040	0.0074	$0.0381 = B_{..W}(X_2)$
$B_{.lB}(X_2)$	0.0178	0.0120	0.0406	$0.0704 = B_{..B}(X_2)$
$B_{.l.}(X_2)$	0.0445	0.0160	0.0480	$0.1085 = B_{...}(X_2)$
$B_{.lW}(X_3)$	0.0345	0.0036	0.0118	$0.0499 = B_{..W}(X_3)$
$B_{.lB}(X_3)$	0.0240	0.0156	0.0419	$0.0815 = B_{..B}(X_3)$
$B_{.l.}(X_3)$	0.0585	0.0192	0.0537	$0.1315\ B_{...}(X_3)$
$B_{.lW}(X_4)$	0.0445	0.0052	0.0118	$0.0615 = B_{..W}(X_4)$
$B_{.lB}(X_4)$	0.0285	0.0177	0.0613	$0.1075 = B_{..B}(X_4)$
$B_{.l.}(X_4)$	00730	0.0229	0.0731	$0.1691 = B_{...}(X_4)$
$B_{.lW}(Y)$	0.1719	0.0219	0.0598	$0.2536 = B_{..W}(Y)$
$B_{.lB}(Y)$	0.1182	0.0782	0.2502	$0.4465 = B_{..B}(Y)$
$B_{.l.}(Y)$	0.2901	0.1001	0.3100	$0.7001 = I(Y)$

The value of the point index $I_{(0.10)}(Y) = 0.7793$ means that the mean of the lower group $\bar{M}_{(0.10).}(Y) = 7270.21$ is equal to the $(1 - 0.7793) \cdot 100\% = 22.06\%$ of the mean $\overset{+}{M}_{(0.10).}(Y) = 32948.33$ of the corresponding upper group, the point index $I_{(0.50)}(Y) = 0.654$ means that the mean $\bar{M}_{(0.50).}(Y) = 15618.21$ is the $(1 - 0.645) \cdot 100\% = 35.0\%$ of $\overset{+}{M}_{(0.50).}(Y) = 45143.31$, and the point index $I_{(0.95)}(Y) = 0.7282$ informs that the mean of the lower group (that contains the 95% of the whole households) is equal to the $(1 - 0.7282) \cdot 100\% = 27.18\%$ of the income mean of the reachest 5% of the households.

7.2.1 *Macroregions Contributions to the Point $I_h(Y)$ and Synthetic $I(Y)$ Indexes*

Now we illustrate the decomposition of the point index $I_{(0.10)}(Y) = 0.7793$ into the three contribution $B_{(0.1)l.}(Y)$ of each macro region:

$$B_{(0.1)1.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h1}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(1|h) = 0.7848 \cdot 0.3383 = 0.2655;$$

$$B_{(0.1)2.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h2}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(2|h) = 0.7707 \cdot 0.1399 = 0.1078;$$

$$B_{(0.1)3.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h3}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(3|h) = 0.7781 \cdot 0.5218 = 0.4061.$$

These values show that the relative variations of the lower means of the three macro-regions w.r.t the upper mean of the whole population are similar, while their relative weights $p(l|h)$ are very different. This explains why there are so remarkable differences among these three contributions. In particular we note that ''the number'' of the households of the South with $Y \leq y_{h(0.10)} = 10600$ Euro is the 52.18% of the ''number'' of the corresponding households of the whole lower group. This explains why the greatest contribution to the point index $I_{(0.10)}(Y) = 0.7793$ comes from the South.

Table 5 shows that the North has the greatest contribution to the the point index $I_{(0.95)}(Y) = 0.7282$, although the relative variation of the lower mean of the South w.r.t the upper mean of the whole population is greater than the one of the North. This happens because the relative weight of the South (0.3344) is smaller than the one of the North (0.4792):

$$B_{(0.95)1.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h1}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(1|h) = 0.7072 \cdot 0.4792 = 0.3389;$$

$$B_{(0.95)2.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h2}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(2|h) = 0.6916 \cdot 0.1864 = 0.1289;$$

$$B_{(0.95)3.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h3}(Y)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(3|h) = 0.7786 \cdot 0.3344 = 0.2604.$$

Table 7 reports for the three macro-regions their:

- relative contributions to the point indexes

$$\beta_{hl.}(Y) = \frac{B_{hl.}(Y)}{I_h(Y)} = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)} \cdot p(l|h), \tag{69}$$

- relative contributions to the synthetic index

$$\beta_{.l.}(Y) = \frac{B_{.l.}(Y)}{I(Y)} = \sum_{h=1}^{r} \beta_{hl.}(Y) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N} \tag{70}$$

- relative weights $n_{.l}/N$.

We note that, the relative contribution of the North increases for increasing values of p; viceversa for the South there is an opposite relation. The contributions to the synthetic inequality index of these two regions are similar. Comparing the relative contribution to the inequality of the macro-regions with their relative weights $n_{.l}/N$ we can assert that the South is a region that increases the income inequality while the contrary happens for the North and the Center.

TABLE 7. - *Subpopulations relative contributions to the point indexes* $\beta_{hl.}(Y) = B_{hl.}(Y)/I_h(Y)$ *and to the synthetic index* $\beta_{.l.}(Y) = B_{.l.}(Y)/I(Y)$

	North	Center	South	
	1	2	3	
$p = 0.10; h = 460$				
$\frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)}$	1.0069	0.9889	0.9984	
$p(l/h)$	0.3383	0.1399	0.5218	
$\beta_{hl.}(Y) = \frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)} \cdot p(l/h)$	0.3407	0.1383	0.5210	1.00
$p = 0.50; h = 3064$				
$\frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)}$	0.9866	0.9718	1.0219	
$p(l/h)$	0.4196	0.1415	0.4388	
$\beta_{hl.}(Y) = \frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)} \cdot p(l/h)$	0.4140	0.1375	0.4484	1.00
$p = 0.95; h = 6841$				
$\frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)}$	0.9712	0.9497	1.0693	
$p(l/h)$	0.4792	0.1864	0.3344	
$\beta_{hl.}(Y) = \frac{\overset{+}{M}_{h.}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)-\bar{M}_{h.}(Y)} \cdot p(l/h)$	0.4654	0.1770	0.3576	1.00
Synthetic index				
$\beta_{.l.}(Y) = \sum_{h=1}^{r} \beta_{hl.}(Y) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N}$	0.4143	0.1429	0.4428	1.00
$n_{.l}/N$	0.4873	0.1886	0.3241	1.00

7.2.2 *Within Part of Subpopulations Contributions and of the Point and Synthetic Indexes*

The last three rows of Tables 3, 4, 5 and 6 report for the point indexes $I_h(Y)$ and the synthetic index $I(Y)$ the joint decompositions by subpopulations and the within and between parts.

Let us consider now, in detail, the within component: of each contribution $B_{(p)l.}(Y)$, and of the three point indexes $I_{(p)}(Y)$, for $p = 0.10, 0.50, 0.95$. The within component $B_{(p)lW}(Y)$ of subpopulation l is equal to the relative difference between its upper and lower mean $\frac{\overset{+}{M}_{hl}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}$ multiplied by the relative frequencies $p(l|h)$ and $a(l|h)$.

For each macroregion Table 8 reports: the calculations for the three within contributions $B_{(p)lW}(Y)$, and the values of the within contributions $B_{.lW}(Y) = \sum_{h=1}^{r} B_{hlW}(Y) \cdot (n_{h.}/N)$.

Table 8 shows that the values of the weights $p(l|h) \cdot a(l|h)$ have a considerable influence on the values of the within components: this explains why the North has the greatest within component. Note that the within component of the point index $I_h(Y)$ is fournished by:

$$B_{h.W}(Y) = \sum_{l=1}^{3} B_{hlW}(Y) = \sum_{l=1}^{3} \frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \cdot p(l|h) \cdot a(l|h).$$

The percentages (shares) of the contribution $B_{hl.}(Y)$ and of the point index $I_h(Y)$, due to the corresponding within components $B_{hlW}(Y)$ and $B_{h.W}(Y)$, are respectively given by:

$$\beta_{hlW}(Y) = B_{hlW}(Y)/B_{hl.}(Y) = \frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)} \cdot a(l|h) \tag{71}$$

and

$$\beta_{h.W}(Y) = B_{h.W}(Y)/I_h(Y) = \sum_{l=1}^{k} \beta_{hlW}(Y) \cdot \beta_{hl.}(Y). \tag{72}$$

Obviously, the last share $\beta_{h.W}(Y)$ is also given by:

$$\beta_{h.W}(Y) = \frac{\sum_{l=1}^{k} \left[\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)\right] p(l|h) \cdot a(l|h)}{\sum_{l=1}^{k} \sum_{g=1}^{k} \left[\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)\right] p(l|h) \cdot a(g|h)}. \tag{73}$$

Finally, the shares of the contribution $B_{.l.}(Y)$ and of the synthetic index $I(Y)$, due to the corresponding within components $B_{.lW}(Y)$ and $B_{..W}(Y)$, are respectively given by:

$$\beta_{.lW}(Y) = B_{.lW}(Y)/B_{.l.}(Y) = \sum_{h=1}^{r} \beta_{hlW}(Y) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N} \tag{74}$$

and

$$\beta_{..W}(Y) = B_{..W}(Y)/I(Y) = \sum_{l=1}^{k} \beta_{.lW}(Y) \cdot \beta_{.l.}(Y). \tag{75}$$

TABLE 8. - *Within part contributions $B_{(p)lW}(Y)$ and $B_{.lW}(Y)$*

		$p(l\|h)$	$a(l\|h)$	$p(l\|h) \cdot a(l\|h)$	$\frac{\overset{+}{M}_{hl}(Y)-\overset{-}{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}$	$B_{hlW}(Y)$
North	$p = 0.10$	0.3383	0.5038	0.1704	0.8630	0.1471
	$p = 0.50$	0.4196	0.5550	0.2329	0.6819	0.1588
	$p = 0.95$	0.4792	0.6411	0.3072	0.7227	0.2220
					$B_{.1W}(Y) =$	0.1719
Center	0.2093	0.1399	0.1940	0.0271	0.8669	0.0235
	0.1281	0.1415	0.2357	0.0334	0.6223	0.0208
	$p = 0.95$	0.1864	0.2308	0.0430	0.5968	0.0257
					$B_{.2W}(Y) =$	0.0219
South	$p = 0.10$	0.5218	0.3021	0.1576	0.5863	0.0924
	$p = 0.50$	0.4388	0.2093	0.0918	0.5862	0.0539
	$p = 0.95$	0.3344	0.1281	0.0427	0.8720	0.0373
					$B_{.3W}(Y) =$	0.0598

By the use of (24) and (29) the last share $\beta_{..W}(Y) = B_{..W}(Y)/I(Y)$ is also given by:

$$\beta_{..W}(Y) = \frac{\sum_{l=1}^{k} \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hl}(Y)-\bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}\right] p(l|h) \cdot a(l|h) \cdot \frac{n_{h.}}{N}}{\sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hg}(X_j)-\bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(l \mid h) \cdot a(g|h) \cdot \frac{n_{h.}}{N}}. \tag{76}$$

Table 9 reports the shares: $\beta_{hlW}(Y)$, $\beta_{h.W}$, $\beta_{.lW}(Y)$ and $\beta_{..W}(Y)$. For the North: the share $\beta_{h1W}(Y)$ increases for increasing values of p and their mean value is $\beta_{.1W}(Y) = 0.5925$. Viceversa, for the South the share $\beta_{h3W}(Y)$ decreases for increasing values of p, and $\beta_{.3W}(Y) = 0.1929$. For the Center $\beta_{.2W}(Y) = 0.2188$. Table 9 informs that the relative frequency $a(l|h)$ has remarkable influence on the share $\beta_{hlW}(Y)$. Moreover, Table 9 shows that the variations of the shares $\beta_{h.W}(Y)$ are smaller than the ones of the North and the South. For the whole country the within component is the 36.22% of the synthetic index $I(Y) = 0.7001$.

TABLE 9. - *Within part shares:* $\beta_{hlW}(Y) = B_{hlW}(Y)/B_{hl.}(Y)$ *of* $B_{hl.}(Y), \beta_{h.W}(Y) = B_{h.W}/I_h(Y)$ *of* $I_h(Y), \beta_{.lW}(Y) = B_{.lW}(Y)/B_{.l.}(Y)$ *of* $B_{.l.}(Y)$ *and* $\beta_{..W}(Y) = B_{..W}/I(Y)$ *of* $I(Y)$

	North	Center	South	Italy
	1	2	3	
$p = 0.10; h = 460$				
$\frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}$	1.0993	1.1248	0.7535	
$a(l/h)$	0.50385	0.194	0.3021	
$\beta_{hlW}(Y) = \frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)} \cdot a(l/h)$	0.554	0.218	0.228	0.338
$p = 0.50; h = 3064$				
$\frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}$	1.0568	0.9791	0.8771	
$a(l/h)$	0.5550	0.2357	0.2093	
$\beta_{hlW}(Y) = \frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)} \cdot a(l/h)$	0.5865	0.2310	0.1837	0.357
$p = 0.95; h = 6841$				
$\frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}$	1.0219	0.8630	1.1199	
$a(l/h)$	0.6411	0.2308	0.1281	
$\beta_{hlW}(Y) = \frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)} \cdot a(l/h)$	0.6551	0.1993	0.1433	0.3913
Synthetic: index				
$\beta_{.lW}(Y) = \sum_{h=1}^{r} \beta_{hlW}(Y) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N}$	0.5925	0.2188	0.1929	0.3622
$n_{.l}/N$	0.4873	0.1886	0.3241	1.00

7.2.3 *Joint Decomposition by Three Macroregions and Four Sources of* $I_h(Y)$ *and* $I(Y)$

The decomposition by sources of the *three* point indexes and of the synthetic index are reported in Tables 10, 11, 12, and 13, respectively.

The relative contribution of the variate X_1 ‘‘payroll income’’ to the point index $I_{(0.1)}(Y)$ is $\beta_{(0.1)..}(X_1) = 0.4382$: this means that the difference between the upper mean $\overset{+}{M}_{(0.1).}(X_1)$ and the lower mean $\bar{M}_{(0.1).}(X_1)$ is the 43.82% of the difference between the corresponding upper and lower means of the total income Y. Table 13 shows that the value of the relative contribution of the variate X_1 to the synthetic index $I(Y)$ is $\beta_{...}(X_1) = 0.4187$: this means that, on average, the difference between the upper mean and the lower mean of X_1 is the 41.57% of the corresponding upper and lower means of Y.

In the case of the joint decomposition of $I_h(Y)$ and of $I(Y)$ it is worth:

- to check if the shares $\beta_{hl.}(X_j) = B_{hl.}(X_j)/B_{hl.}(Y)$ are influenced by the subpopulations, and by the cumulative frequency p;
- to find some relationship between the share $\beta_{h..}(X_j)$ and the k shares $\beta_{hl.}(X_j)$, as well as between the share $\beta_{...}(X_j)$ and the k shares $\beta_{.l.}(X_j)$.

From (20) and (48) we obtain:

$$\beta_{hl.}(X_j) = \frac{B_{hl.}(X_j)}{B_{hl.}(Y)} = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}. \tag{77}$$

TABLE 10. - *Shares $\beta_{(0.1)l.}(X_j)$ of income sources to the region contributions $B_{(0.1)l.}(Y)$ and shares $\beta_{(0.1)..}(X_j)$ of income sources to the point index $I_{(0.1)}(Y)$*

$p = 0.10$; $h = 460$	North	Center	South	Italy
	$l = 1$	$l = 2$	$l = 3$	
$\beta_{(0.1)l.}(X_1)$	0.4045	0.4397	0.4597	$0.4382 = \beta_{(0.1)..}(X_1)$
$\beta_{(0.1)l.}(X_2)$	0.2222	0.2199	0.1906	$0.2056 = \beta_{(0.1)..}(X_2)$
$\beta_{(0.1)l.}(X_3)$	0.1318	0.1160	0.1340	$0.1308 = \beta_{(0.1)..}(X_3)$
$\beta_{(0.1)l.}(X_4)$	0.2411	0.2245	0.2155	$0.2255 = \beta_{(0.1)..}(X_4)$
	1.0000	1.0000	1.000	1.0000

TABLE 11. - *Shares $\beta_{(0.5)l.}(X_j)$ of income sources to the region contributions $B_{(0.5)l.}(Y)$ and shares $\beta_{(0.5)..}(X_j)$ of income sources to the point index $I_{(0.5)}(Y)$*

$p = 0.50$; $h = 3064$	North	Center	South	Italy
	$l = 1$	$l = 2$	$l = 3$	
$\beta_{(0.5)l.}(X_1)$	0.4414	0.4722	0.4671	$0.4570 = \beta_{(0.5)..}(X_1)$
$\beta_{(0.5)l.}(X_2)$	0.1426	0.1478	0.1425	$0.1431 = \beta_{(0.5)..}(X_2)$
$\beta_{(0.5)l.}(X_3)$	0.1795	0.1733	0.1589	$0.1696 = \beta_{(0.5)..}(X_3)$
$\beta_{(0.5)l.}(X_4)$	0.2368	0.2067	0.2315	$0.2303 = \beta_{(0.5)..}(X_4)$
	1.0000	1.0000	1.000	1.0000

TABLE 12. - *Shares* $\beta_{(0.95)l.}(X_j)$ *of income sources to the region contributions* $B_{(0.95)l.}(Y)$ *and shares* $\beta_{(0.95)..}(X_j)$ *of income sources to the point index* $I_{(0.95)}(Y)$

$p = 0.95$; $h = 6841$	North	Center	South	Italy
	$l = 1$	$l = 2$	$l = 3$	
$\beta_{(0.95)l.}(X_1)$.	0.2721	0.2901	0.2919	$0.2823 = \beta_{(0.95)..}(X_1)$
$\beta_{(0.95)l.}(X_2)$	0.1228	0.1202	0.1310	$0.1252 = \beta_{(0.95)..}(X_2)$
$\beta_{(0.95)l.}(X_3)$	0.3237	0.3196	0.2980	$0.3138 = \beta_{(0.95)..}(X_3)$
$\beta_{(0.95)l.}(X_4)$	0.2815	0.2700	0.2788	$0.2785 = \beta_{(0.95)..}(X_4)$
	1.0000	1.0000	1.000	1.0000

TABLE 13. - *Relative contributions*: $\beta_{...}(X_j)$; $\beta_{.l.}(X_j)$

	North	Center	South	Italy
	$l = 1$	$l = 2$	$l = 3$	
$\beta_{.l.}(X_1)$	0.3930	0.4195	0.4355	$0.4157 = \beta_{...}(X_1)$
$\beta_{.l.}(X_2)$	0.1563	0.1598	0.1548	$0.1550 = \beta_{...}(X_2)$
$\beta_{.l.}(X_3)$	0.2017	0.1918	0.1732	$0.1878 = \beta_{...}(X_3)$
$\beta_{.l.}(X_4)$	0.2516	0.2288	0.2358	$0.2415 = \beta_{...}(X_4)$
	1.0000	1.0000	1.000	1.0000

Now, the use of relations $B_{h..}(X_j) = \sum_{l=1}^{k} B_{hl.}(X_j)$ and $I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y)$ in (61) fournishes:

$$\beta_{h..}(X_j) = \frac{B_{h..}(X_j)}{I_h(Y)} =$$

$$= \frac{\sum_{l=1}^{k} B_{hl.}(X_j)}{\sum_{l=1}^{k} B_{hl.}(Y)} = \frac{1}{I_h(Y)} \cdot \sum_{l=1}^{k} \frac{B_{hl.}(X_j)}{B_{hl.}(Y)} B_{hl.}(Y).$$

Thus,

$$\beta_{h..}(X_j) = \sum_{l=1}^{k} \frac{B_{hl.}(X_j)}{B_{hl.}(Y)} \cdot \frac{B_{hl.}(Y)}{I_h(Y)} = \sum_{l=1}^{k} \beta_{hl.}(X_j) \cdot \beta_{hl.}(Y). \tag{78}$$

In other words, the share $\beta_{h..}(X_j)$ is the weighted mean of the k shares $\beta_{hl.}(X_j)$ with weights $\beta_{hl.}(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)} \cdot p(l|h)$.

For the synthetic index the analogous of (78) is:

$$\beta_{...}(X_j) = \sum_{l=1}^{k} \frac{B_{.l.}(X_j)}{B_{.l.}(Y)} \cdot \frac{B_{.l.}(Y)}{I(Y)} = \sum_{l=1}^{k} \beta_{.l.}(X_j) \cdot \beta_{.l.}(Y). \tag{79}$$

The values of the shares $\beta_{h..}(X_j)$ and $\beta_{hl.}(X_j)$ reported in Tables 10, 11 and 12 are consistent with relation (78), the values of the share $\beta_{...}(X_j)$ and of the shares $\beta_{.l.}(X_j)$ reported in Table 13 are coherent with relation (79). Beside that, the shares $\beta_{.l.}(X_j)$ of the three macroregions are very similar and consequently their differences with the corresponding relative contributions $\beta_{...}(X_j)$ of the whole Country are negligible particularly for the variate X_2 (Pensions and net transfers). We end this section observing that the values of the relative contributions $\beta_{...}(X_j)$ evaluated in this paper (data supplied by the 2012 Central Bank of Italy sample survey) are very similar to the ones obtained in Zenga (2012), regarding the 2008 Central Bank of Italy survey.

7.3 *Decomposition by Sources for Each Italian Macro-regions*

Table 14 reports for $p = 0.10$, 0.50 and 0.95 the decomposition by sources of the point indexes $I_{hl}(Y)$ and $I_h(Y)$, while Table 15 reports the decomposition by sources of the synthetic indexes $I_{.l}(Y)$ and $I(Y)$.

TABLE 14. - *Decomposition by sources of the point indexes $I_{hl}(Y)$ and $I_h(Y)$*

	North		Center		South		Italy	
	$B_{h1}(\cdot)$	$\beta_{h1}(\cdot)$	$B_{h2}(\cdot)$	$\beta_{h2}(\cdot)$	$B_{h3}(\cdot)$	$\beta_{h3}(\cdot)$	$B_{h..}(\cdot)$	$\beta_{h..}(\cdot)$
$p = 0.10; h = 460$								
X_1	0.3269	*0.4085*	0.3270	*0.4135*	0.3550	*0.4894*	0.3415	*0.4382*
X_2	0.1761	*0.2200*	0.1774	*0.2243*	0.1321	*0.1820*	0.1602	*0.2056*
X_3	0.1040	*0.1300*	0.0890	*0.1125*	0.1046	*0.1442*	0.1019	*0.1308*
X_4	0.1933	*0.2415*	0.1975	*0.2497*	0.1338	*0.1844*	0.1757	*0.2255*
$I_{hl}(Y)$	0.8003	*1.0000*	0.7909	*1.0000*	0.7255	*1.0000*	0.7793	*1.0000*
$p = 0.50; h = 3064$								
X_1	0.2842	*0.4320*	0.2766	*0.4386*	0.3447	*0.5397*	0.2989	*0.4570*
X_2	0.0991	*0.1506*	0.1077	*0.1707*	0.0576	*0.0903*	0.0936	*0.1431*
X_3	0.1163	*0.1767*	0.1021	*0.1618*	0.1144	*0.1791*	0.1109	*0.1696*
X_4	0.1583	*0.2406*	0.1444	*0.2289*	0.1220	*0.1910*	0.1506	*0.2303*
$I_{hl}(Y)$	0.6578	*1.0000*	0.6307	*1.0000*	0.6387	*1.0000*	0.6540	*1.0000*
$p = 0.95; h = 6841$								
X_1	0.1781	*0.2502*	0.2640	*0.4004*	0.1978	*0.2480*	0.2056	*0.2823*
X_2	0.0949	*0.1334*	0.0956	*0.1450*	0.0443	*0.0555*	0.0912	*0.1252*
X_3	0.2317	*0.3255*	0.1389	*0.2107*	0.3401	*0.4264*	0.2285	*0.3138*
X_4	0.2070	*0.2909*	0.1608	*0.2439*	0.2154	*0.2700*	0.2028	*0.2785*
$I_{hl}(Y)$	0.7117	*1.0000*	0.6593	*1.0000*	0.7975	*1.0000*	0.7282	*1.0000*

Table 14 shows that there are remarkable differences between the relative contributions $\beta_{hl}(X_j)$ and $\beta_{h..}(X_j)$. There are also considerable differences between the contributions $\beta_{hl}(X_j)$ and the contributions $\beta_{hl.}(X_j)$ reported in Tables 10, 11 and 12. Indeed it is possible to show that the relative contributions $\beta_{hl}(X_j)$ are equal to the relative contributions

$$\beta_{hlW}(X_j) = B_{hlW}(X_j)/B_{hlW}(Y).$$

Note that the contributions $B_{hlW}(X_j)$ and $B_{hlW}(Y)$ are reported in Tables 3, 4 and 5.

TABLE 15. - *Decomposition by sources of the synthetic indexes $I_{.l}(Y)$ and $I(Y)$*

	North		Center		South		Italy	
	$B_{.1}(\cdot)$	$\beta_{.1}(\cdot)$	$B_{.2}(\cdot)$	$\beta_{.2}(\cdot)$	$B_{.3}(\cdot)$	$\beta_{.3}(\cdot)$	$B_{...}(\cdot)$	$\beta_{...}(\cdot)$
X_1	0.2673	*0.3847*	0.2725	*0.4134*	*0.3370*	0.4871	0.2910	*0.4157*
X_2	0.1074	*0.1546*	0.1173	*0.1779*	*0.0892*	0.1289	0.1085	*0.1550*
X_3	0.1400	*0.2015*	0.1121	*0.1701*	*0.1309*	0.1892	0.1315	*0.1878*
X_4	0.1803	*0.2594*	0.1573	*0.2386*	*0.1348*	0.1948	0.1691	*0.2415*
$I_{.l}(Y)$	0.6949	*1.0000*	0.6592	*1.0000*	*0.6919*	1.0000	0.7001	*1.0000*

8. PRINCIPAL RESULTS AND CONCLUSIONS

$X_1, X_2, \ldots, X_c$ are c variates (income sources) observable on each of the N units of a finite population, and Y (total income) is the sum $\sum_{j=1}^{c} X_j$. The N units are partitioned in k different subpopulations; $n_{.l}$ is the frequency of subpopulation l : $\sum_{l=1}^{k} n_{.l} = N$. The set of the distinct values assumed by Y is $\{y_1 < \ldots < y_h < \ldots < y_r\}$ and $\{n_{1.}, \ldots, n_{h.}, \ldots, n_{r.}; \sum_{h=1}^{r} n_{h.} = N\}$ are the corresponding frequencies. The number of units of subpopulation l with $Y = y_h$ is n_{hl}. The Zenga (2007) point inequality index $I_h(Y)$ is the relative variation of the mean $\bar{M}_{h.}(Y)$ of the lower group $(Y \leq y_h)$ w.r.t. the mean $\overset{+}{M}_{h.}(Y)$ of the upper group $(Y > y_h)$: $I_h(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}{\overset{+}{M}_{h.}(Y)}$. The synthetic index $I(Y)$ is the mean of $I_h(Y)$: $I(Y) = M(I_h(Y)) = \sum_{h=1}^{r} I_h(Y) \cdot \frac{n_{h.}}{N}$.

The decompositions of $I(Y)$ by subpopulations (Radaelli 2008, 2010; Zenga 2016) and by sources (Zenga *et al.* 2012) are based on a two-step approach. In the first step additive decompositions of $I_h(Y)$ are obtained, and, then (second step) putting these decompositions in $I(Y) = M(I_h(Y))$ the corresponding decompositions of $I(Y)$ are derived.

The Zenga *et al.* (2012) decomposition by sources of $I_h(Y)$ is given by (first step)

a) $I_h(Y) = \sum_{j=1}^{c} B_{h..}(X_j)$; $B_{h..}(X_j) = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y)}$,

where $\bar{M}_{h.}(X_j)$ and $\overset{+}{M}_{h.}(X_j)$ are the means of X_j in the lower and upper groups.
The decomposition by sources of $I(Y)$ is given by (second step)

$b)$ $I(Y) = \sum_{j=1}^{c} B_{...}(X_j); B_{...}(X_j) = \sum_{h=1}^{r} B_{h..}(X_j) \cdot \frac{n_{h.}}{N}$.
For $I_h(Y)$, Zenga (2016) has obtained the following $k \times k$ decomposition by subpopulations (first step)

$c)$ $I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{hlg}(Y); B_{hlg}(Y) = \frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} p(l|h) \cdot a(g|h)$,
where $\overset{+}{M}_{hg}(Y)$ and $\bar{M}_{hl}(Y)$ are the upper and lower means in the subpopulations g and l , and $a(g|h)$ and $p(l|h)$ are their relative frequencies. The decomposition by subpopulations of $I(Y)$ is given by (second step)

$d)$ $I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{.lg}(Y);\ B_{.lg}(Y) = \sum_{h=1}^{r} B_{hlg}(Y) \cdot \frac{n_{h.}}{N}$.
In the present paper, using the relations $\bar{M}_{hl}(Y) = \sum_{j=1}^{c} \bar{M}_{hl}(X_j)$ and $\overset{+}{M}_{hg}(Y) = \sum_{j=1}^{c} \overset{+}{M}_{hg}(X_j)$ in decomposition $c)$, the following $k \times k \times c$ joint decomposition of $I_h(Y)$ is obtained (first step):

$e)$ $I_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{hlg}(X_j); B_{hlg}(X_j) = \frac{\overset{+}{M}_{hg}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)} \cdot p(l|h) \cdot a(g|h)$.
The joint decomposition of $I(Y)$ is given by (second step)

$f)$ $I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \sum_{j=1}^{c} B_{.lg}(X_j);\ B_{.lg}(X_j) = \sum_{h=1}^{r} B_{hlg}(X_j) \cdot \frac{n_{h.}}{N}$.

Following the two step approach, Zenga (2013) has decomposed by sources the Gini (1914) and the Bonferroni (1930) point and synthetic indexes, and he has also shown that the contribution of X_j to the synthetic Gini index obtained with the two step approach, is ''equal'' to those obtained by Rao (1969), by Lerman and Yitzhaki (1984,1985), and by Radaelli and Zenga (2002, 2005). Unfortunately, these latter decompositions of the Gini index can't give informations on the contribution of the component X_j to the Gini point index. This happens because these three decompositions are based on the following expression of the synthetic Gini index: $\tilde{G}(Y) = \Delta(Y)/(2 \cdot M(Y))$.

Another important result is that, from the decomposition by subpopulations and from the joint decomposition, it is possible to derive others interesting additive decompositions of $I_h(Y)$ and $I(Y)$.

Thus, from $c)$ the following decompositions $g)$ and $h)$ are obtained.

$g)$ $I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y);\ B_{hl.}(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \cdot p(l|h)$.

Note that $B_{hl.}(Y)$ is the product of the relative variation $\left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)}\right]$ of the lower mean $\bar{M}_{hl}(Y)$ w.r.t. the upper mean $\overset{+}{M}_{h.}(Y)$ with the relative frequency $p(l|h)$. Consequently, $B_{hl.}(Y)$ can be interpreted as the contribution of the subpopulation l to $I_h(Y)$.

$h)$ $I_h(Y) = B_{h.W}(Y) + B_{h.B}(Y)$,
where $B_{h.W}(Y)$ and $B_{h.B}(Y)$ are the within and the between parts of $I_h(Y)$.

By the usual procedure, the decompositions $g)$ and $h)$ have been extended to $I(Y)$.

$i)$ $I(Y) = \sum_{l=1}^{k} B_{.l.}(Y)$,

$l)$ $I(Y) = B_{..W}(Y) + B_{..B}(Y)$,

where $B_{..W}(Y)$ and $B_{..B}(Y)$ are the within and the between parts of the synthetic index $I(Y)$, respectively. Note that in $B_{..W}(Y)$ only comparisons between upper and lower means of the same subpopulations are involved, while in $B_{..B}(Y)$ only comparisons between upper and lower means of different subpopulations are involved.

It is also important to point out that, from the joint decomposition we can derive the decomposition by subpopulations and the decomposition by sources. In particular:

$B_{hlg}(Y) = \sum_{j=1}^{c} B_{hlg}(X_j)$;

$B_{h..}(X_j) = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{hlg}(X_j) = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y)} = B_h(X_j)$, and

$B_{...}(X_j) = \sum_{h=1}^{r} B_{h..}(X_j) \cdot \frac{n_{h.}}{N} = B(X_j)$.

Moreover, $B_{hl.}(X_j) = \sum_{g=1}^{k} B_{hlg}(X_j) = \left[\frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y)}\right] \cdot p(l|h)$, and

$\sum_{j=1}^{c} B_{hl.}(X_j) = B_{hl.}(Y)$. In other words, $B_{hl.}(X_j)$ is the ‘‘additive’’ contribution of X_j to $B_{hl.}(Y)$.

The theoretical results of this paper are applied to the 2012 Bank of Italy sample survey on household income and wealth. Section 7 illustrates the decompositions: of the point index $I_{h(p)}(Y) = I_{(p)}(Y)$ for three values (0.10; 0.50; 0.95) of the cumulative relative frequency p, and of the synthetic index $I(Y)$. Now it is worth to remark that, the interpretation of the decompositions illustrated above can be facilitated by the calculation of some relative contributions.

In the case of the decomposition by sources, the relative contributions of X_j to $I_h(Y)$ and to $I(Y)$ are:

$$\beta_{h..}(X_j) = \frac{B_{h..}(X_j)}{I_h(Y)} = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{h.}(X_j)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}$$

and

$$\beta_{...}(X_j) = \frac{B_{...}(X_j)}{I(Y)} = \sum_{h=1}^{r} \beta_{h..}(X_j) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N}.$$

Moreover these relative contributions should be compared with the shares $\gamma_{..}(X_j) = \frac{M(X_j)}{M(Y)}$ in order to discern whether a given income sources X_j has an exacerbating or a mitigating impact on inequality in the distribution of total income Y.

Table 16 shows that the relative contributions of X_2 (Pensions and net transfers) are smaller than the corresponding share $\gamma_{..}(X_2) = 0.275$. This means that X_2 decreases the inequality.

TABLE 16. - *Relative contributions of* X_2

	$B_{(p)..}(X_2)$	$I_{(p)}$	$\beta_{(p)..}(X_2)$	$\gamma_{..}(X_2)$
$p = 0.10;\ h = 460$	0.1602	0.7793	0.206	0.275
$p = 0.50;\ h = 3064$	0.0936	0.6540	0.143	
$p = 0.95;\ h = 6841$	0.0912	0.7282	0.125	
	$B_{...}(X_2) = 0.109$	$I(Y) = 0.700$	$\beta_{...}(X_2) = 0.155$	

the case of the decomposition by subpopulations, the relative contributions of the subpopulation l to $I_h(Y)$ and to $I(Y)$ are:

$$\beta_{hl.}(Y) = \frac{B_{hl.}(Y)}{I_h(Y)} = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)} \cdot p(l|h) \text{ and}$$

$$\beta_{.l.}(Y) = \frac{B_{.l.}(Y)}{I(Y)} = \sum_{h=1}^{r} \beta_{hl.}(Y) \cdot \frac{I_h(Y) \cdot n_{h.}}{I(Y) \cdot N}.$$

Now we remark that, in the case of independence in distribution between Y and the subpopulations, $\beta_{hl.}(Y) = \beta_{.l.}(Y) = n_{.l}/N$. Thus, it may be useful to compare $\beta_{hl.}(Y)$ and $\beta_{.l.}(Y)$ with $n_{.l}/N$ (see Table 7).

Finally, in the case of the joint decomposition of $I_h(Y)$ are useful the k shares $\beta_{hl.}(X_j) = \frac{B_{hl.}(X_j)}{B_{hl.}(Y)} = \frac{\overset{+}{M}_{h.}(X_j) - \bar{M}_{hl}(X_j)}{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}$ and their relationship with $\beta_{h..}(X_j)$: $\beta_{h..}(X_j) = \sum_{l=1}^{k} \beta_{hl.}(X_j) \cdot \beta_{hl.}(Y)$. While in the case of the joint decomposition of the synthetic index are useful the k shares $\beta_{.l.}(X_j) = \frac{B_{.l.}(X_j)}{B_{.l.}(Y)}$ and their relationship with $\beta_{...}(X_j) = B_{...}(X_j)/I(Y) = \sum_{l=1}^{k} \beta_{.l.}(X_j) \cdot \beta_{.l.}(Y)$.

The values of $\beta_{h..}(X_j)$ and of $\beta_{hl.}(X_j)$, reported in tables 10, 11 and 12, are ''consistent'' with their relationship, as well as the values of $\beta_{...}(X_j)$ and of $\beta_{.l.}(X_j)$, reported in Table 13, are coherent with their relashonship. Beside that, the shares $\beta_{.l.}(X_j)$ of the three Italians macroregions are very similar and consequently their differences with the corresponding relative contributions $\beta_{...}(X_j)$ of the whole Country are negligible particularly for the variate X_2 (Pensions and net transfers).

We end this paper pointing out that from Tables 3, 4 and 5 it is possible to obtain other useful information. In particular: a) Table 3 shows the remarkable role played by the payrol incomes of the South in explaining inequality at the botton of total income distribution ($p = 0.1$); b) Table 5 shows that self-employment and property incomes of the North play a major role in determining inequality at the top of total income distribution ($p = 0.95$). More specifically, this effect is mainly caused by the within-group disparities in the North and by the between-group disparities between the North and the other regions, while the within-group inequality contributions from the Center and the South are far less important.

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