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DECOMPOSITION BY SUBPOPULATIONS OF GINI, BONFERRONI AND ZENGA INEQUALITY MEASURES

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SUMMARY

This paper presents a common framework for the decompositions by subpopulations of Gini, Bonferroni and Zenga synthetic inequality measures. These three synthetic indexes are the weighted arithmetic means of the corresponding point measures and applying the Zenga two-step approach, decompositions based on means comparison are obtained. In the first step additive decompositions are derived for the point indexes and in the second step, using the decompositions of the point measures, we obtain the decompositions by subpopulations of the synthetic indexes. In particular the point and the synthetic indexes are decomposed in the sum of subpopulations contributions which in turn are decomposed in within and between components. The decompositions obtained can be utilized in the case of non-overlapping subpopulations as well as in the overlapping case and in the present work two numerical examples are illustrated to pointing out that whereas it is possible to obtain negative contributions for the Gini and the Bonferroni indexes, this cannot happen for the Zenga index.

Keywords: Gini Index, Bonferroni Index, Zenga Index, ''two-step'' Approach, Decomposition by Subpopulations, Decomposition by Sources, Joint Decomposition by Subpopulations and Sources.
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1. INTRODUCTION

A very important characteristic of an inequality measure is its suitability in the decompositions by subpopulations and by sources, as well as its joint decomposition by subpopulations and sources (Rao, 1969; Mehran, 1975; Shorroks, 1980; Tarsitano, 1990; Silber, 1989; Dagum, 1997a, 1997b; Zenga, 2001; Ebert, 2010; Frosini, 2012). The aim of all the above mentioned papers is the decomposition (by sources, by subpopulations) of synthetic inequality indexes: Gini, Bonferroni, Herfindahl, Pietra-Ricci and Theil.

On the other hand, the decompositions of the Zenga (2007) $I(Y)$ synthetic inequality measure, by subpopulations (Radaelli, 2008, 2010; Zenga, 2016a), and by sources (Zenga, Radaelli, Zenga, 2012) have been obtained by the use of a two- step approach. In particular: additive decompositions of the Zenga point index $I_h(Y)$ are

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obtained in the first step, and, then (second step) by averaging these decompositions, they obtain the corresponding decompositions of $I(Y)$.

Recently, Zenga (2013) has used this two step approach for the decomposition by sources of the Gini $G(Y)$ and Bonferroni $V(Y)$ indexes, too. In addition, Pasquazzi and Zenga (2018) applied this approach to assess the factor component contributions to the Gini, Bonferroni and Zenga inequality idexes. In this latter application income data from Eurostat's European Community Household Panel Survey (ECHP) were utilized.

It is important to point out that, Zenga and Valli (2016), and Zenga (2016b) have utilized the ''two-step'' approach for the decompositions by subpopulations of the Bonferroni and the Gini indexes, too.

Moreover, by the use of the ''two step'' approach Zenga (2015), Zenga and Valli (2017, 2018) have obtained respectively the joint decompositions by subpopulations and sources of the Zenga, Bonferroni and Gini indexes.

Note that the decompositions by k subpopulations proposed before 2016 usually follow a similar scheme: the synthetic index is decomposed into a within and a between term. The decompositions proposed by Zenga (2016a, 2016b) and Zenga and Valli (2016) decompose, first of all, the point measures in the sum of k terms, one for each subpopulation. In addition, each of these terms can be split into a within and a between component. In the second step these decompositions are extended to the corresponding synthetic measures. Note that the decompositions obtained can be utilized in the case of non-overlapping subpopulations as well as in the overlapping case. The rest of the paper is organized as follows. In Section 2 we give some definitions and notation regarding the case that a non negative variate Y is observed on the N units of a finite population and that these units can be partitioned, according to some relevant characteristic, into k different subpopulations whose sizes are denoted by $n_{.g}(g = 1, ..., k)$. The N values of Y arranged in non decreasing order are $\{0 \le y_{(1)} \le y_{(2)} \le \ldots \le y_{(i)} \le y_{(N)} > 0\}$, while $\{0 \le y_1 < \ldots < y_h < \ldots < y_r\}$ is the set of the distinct values assumed by Y over the k subpopulations and $\{n_{1.}, \ldots, n_{h.}, \ldots, n_{r.}; \sum_{h=1}^{r} n_{h.} = N\}$ are the corresponding frequencies. Moreover, Section 2 reports (Table 1) the $r \times k$ bivariate distribution of the N units according to the k subpopulations and the r distinct values assumed by Y. In particular: n_{hg} denotes the frequency of y_h in the subpopulation g, $n_{h.} = \sum_{g=1}^{k} n_{hg}$ is the frequency of y_h in the whole population and $n_{.g}$ is the size of subpopulation g. This bivariate distribution plays a significant role in the decomposition of the three indexes. Moreover, in Section 2 we introduce the Gini (1914) and the Bonferroni (1930) point and synthetic measures, and recall that, as pointed out by De Vergottini (1940), the Bonferroni synthetic index is equal to the unweighted average of the Gini point measures. Note that Gini and Bonferroni defined their indexes using the N values $y_{(i)}$. Then, we illustrate the Zenga (2007) point $I_h(Y)$ and synthetic $I(Y)$ indexes. In particular $I_h(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}{\overset{+}{M}_{h.}(Y)}$, where $\bar{M}_{h.}(Y)$ is the mean (lower mean) of Y computed on the $P_{h.}$ units of the lower group $\{Y \le y_h\}$ and $\overset{+}{M}_{h.}(Y)$ is the mean (upper mean) of the $(N - P_{h.})$ units of the corresponding upper group $\{Y > y_h\}$. $I(Y)$ is given by

$I(Y) = \sum_{h=1}^{r} I_h(Y) \cdot (n_{h.}/N)$. Section 2 also provides ‘‘convenient’’ expressions of the Bonferroni and Gini indexes in the case of frequency distribution framework. In Section 3 we introduce some additional definitions and notation. For the distribution $\{(y_h, n_{hg}) : h = 1, ..., r\}$ of the subpopulation g: $P_{hg} = \sum_{t=1}^{h} n_{tg}$ and $(n_{.g} - P_{hg})$ are the units of subpopulation g with $\{Y \leq y_h\}$ and $\{Y > y_h\}$ respectively, and, $\bar{M}_{hg}(Y)$ and $\overset{+}{M}_{hg}$ are the corresponding lower and upper means. The decompositions by subpopulations of the Zenga, Bonferroni and Gini point indexes are based on the relations between the means $[M(Y), \bar{M}_{h.}(Y), \overset{+}{M}_{h.}(Y)]$ of the whole population and the corresponding means $[M_l(Y), \bar{M}_{hl}(Y), \overset{+}{M}_{hl}(Y)]$ of the k subpopulations. In particular, in the first step we decompose the Zenga I_h, the Bonferroni V_h, and the Gini G_h point indexes in the sum of k contributions, one for each subpopulation: $I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y)$, $V_h(Y) = \sum_{l=1}^{k} V_{hl.}(Y)$, $G_h(Y) = \sum_{l=1}^{k} C_{hl.}(Y)$. In Section 3 we obtain also the within and the between components of these contributions $(B_{hl.}, V_{hl.}\, C_{hl.})$, and, consequently the within and the between components of the three inequality point measures:

$$I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y) = \sum_{l=1}^{k} (B_{hlW} + B_{hlB}) = B_{h.W} + B_{h.B};$$

$$V_h(Y) = \sum_{l=1}^{k} V_{hl.}(Y) = \sum_{l=1}^{k} (V_{hlW} + V_{hlB}) = V_{h.W} + V_{h.B};$$

$$G_h(Y) = \sum_{l=1}^{k} C_{hl.}(Y) = \sum_{l=1}^{k} (C_{hlW} + C_{hlB}) = C_{h.W} + C_{h.B}.$$

Then (second step), we extend these decompositions to the three synthetic indexes $I(Y)$, $V(Y)$, and $G(Y)$. In Section 4 we report in the three $r \times k$ tables (Tables 10, 11, 12) some of the results obtained in Section 3. In each cell hl, $(h = 1, ..., r;\ l = 1, ..., k)$, of Table 12 we report the product $\frac{n_{h.}}{N} B_{hl.}$. The sum of these $r \times k$ products $\sum_{h=1}^{r} \sum_{l=1}^{k} \frac{n_{h.}}{N} B_{hl.}(Y)$ gives the synthetic Zenga $I(Y)$ index. Moreover, the sum $\sum_{l=1}^{k} \frac{n_{h.}}{N} B_{hl.}(Y)$ of the values reported in the l cells of row h gives $\frac{n_{h.}}{N} I_h(Y)$. These r products are reported in the ‘‘marginal’’ column of Table 12, and obviously, their sum gives $I(Y)$. In addition, the sum of the r values reported in the r cells of column l gives $\sum_{h=1}^{r} \frac{n_{h.}}{N} B_{hl.}(Y) = B_{.l.}$. These k values are reported in the ‘‘marginal’’ row; $\sum_{l=1}^{k} B_{.l.}(Y) = I(Y)$, and $B_{.l.}(Y)$ is the contribution of subpopulation l to $I(Y)$. Note that the value of $I(Y)$ is equal to the sum of the areas of the r rectangles, with bases $\frac{n_{h.}}{N}$ and heights $I_h(Y)$, related to the inequality diagram of the point inequality measure $I_h(Y)$. In addition, the area of the h^{th} rectangle can be split into the areas of l sub-rectangles with (common) base $\frac{n_{h.}}{N}$ and heights $B_{hl.}$, $(l = 1, ..., k)$. Thus, $\frac{n_{h.}}{N} B_{hl.}$ can be interpreted as the contribution of the subpopulation l to the contribution $\frac{n_{h.}}{N} I_h(Y)$. This graphical representation can be extended to the Bonferroni and the Gini indexes, too. In Section 4.1 we illustrate, (Tables 3, 4), the calculation of the point and the synthetic measures of the three indexes in the case of a population of 20 units partitioned into 5 non-overlapping subpopulations with sizes $n_{.l} = 4$, $(l = 1, ..., 5)$. Thus, for each subpopulation we have reported (Tables: 5, 6, 7, 8, 9) all the values necessary for the calculation of the contributions reported in Tables 10, 11, and 12, and for the calculation of the within and between parts of $B_{hl.}$, $V_{hl.}$ and $C_{hl.}$. In applied analysis, to obtain informations on the decompositions, it is useful to aggregate the N units in groups delimited by quantiles. Using the quintiles the five groups of the numerical examples at hand are: $(y_{1+4\cdot(s-1)}, ..., y_{4\cdot s})$, $s = 1, \ldots, 5$.

These groups are the $k = 5$ subpopulations of the numerical example of the Section 4.1. In the case of the $r \times k$ bivariate Table 11, each h^{th} row reports the five contributions $\frac{n_{h.}}{20} \cdot V_{hl.}$ and the marginal contribution $\frac{n_{h.}}{20} \cdot V_{h..} = \frac{n_{h.}}{20} \cdot V_h$. These six contributions are related to the interval $D_h = (p_{h-1.} \dashv p_{h.})$ of the Bonferroni graph. In the case of the ''quintiles'' we need to aggregate the first 4 rows $(h = 1, ..., 4)$, the second 4 rows $(h = 5, ..., 8)$, and so on. In this way we obtain contributions that are related to the intervals: $D_{(1)} = (0 \dashv 0.2)$, $D_{(2)} = (0.2 \dashv 0.4)$, ..., $D_{(5)} = (0.8 \dashv 1.0)$. This means that we can reppresent the value of $V(Y)$ as the sum of the areas of 5 rectangles with bases 0.2 and heights

$$\overline{V}_s = \frac{1}{4} \cdot \sum_{h=1+4(s-1)}^{4s} V_h \, , s = 1, ..., 5.$$

By the ''aggregations'' of the rows of the Tables 10, 11 and 12 we have obtained the decompositions of the three inequality measures reported in Tables 13, 14, and 15. These three latter Tables report also the within and the between parts of the ''contributions'' and of the synthetic measures. In Section 4.2 we illustrate, (Tables 17, 18, 19), the calculation of the point and the synthetic measures of the three indexes of a population of $N = 7$ units partitioned into $k = 3$ overlapping subpopulations with sizes: $n_{.1} = 2$, $n_{.2} = 3$, and $n_{.3} = 2$. Then for each subpopulation we have reported (Tables 17, 18, 19, 20) all the values necessary for the calculation of the ''contributions'' of the decompositions by subpopulations of the three inequality measures (Tables 21, 22, 23). Section 5 is devoted to the principal results and to the conclusions.

2. DEFINITIONS AND NOTATION

Let Y denote a non negative variate, usually income, observed on the N units of a finite population. The N units can be partitioned, according to some relevant characteristic, into k different subpopulations whose sizes are denoted by $n_{.g}$ $(g = 1, ..., k)$. Let: $0 \leq y_{(1)} \leq ... \leq y_{(i)} \leq ... \leq y_{(N)} > 0$ be the N values of Y arranged in non- decreasing order; $Q_{(i)}(Y) = \sum_{t=1}^{i} y_{(t)}$,(i = 1, ..., N) be the income of the i poorest population units; $T = Q_{(N)}(Y)$; $M = M(Y) = \frac{T}{N}$; $\bar{M}_{(i)}(Y) = \frac{Q_{(i)}(Y)}{i}, (i = 1, \ldots, N)$. According to Gini (1914) the relative inequality, corresponding to the relative frequency $p_{(i)} = \frac{i}{N}$, is given by

$$\rho_{(\frac{i}{N})} = \frac{p_{(i)} - q_{(i)}}{p_{(i)}}, \quad (i = 1, ..., N),$$

where

$$q_{(i)} = \frac{Q_{(i)}}{T} = \frac{i}{N} \times \frac{\bar{M}_{(i)}}{M} = p_{(i)} \times \frac{\bar{M}_{(i)}}{M},$$

is the ordinate of the Lorenz (1905) curve. Note that for $i = N, p_{(N)} = q_{(N)} = 1$, and $\frac{p_{(N)}-q_{(N)}}{q_{(N)}} = 0$. The synthetic inequality index $\tilde{G}(Y)$ proposed by Gini(1914) is the weighted mean of $\rho_{(\frac{i}{N})}$with weights $p_{(i)}$:

$$\tilde{G}(Y) = \frac{1}{\sum_{i=1}^{N-1} p_{(i)}} \cdot \sum_{i=1}^{N-1} \frac{p_{(i)} - q_{(i)}}{p_{(i)}} \cdot p_{(i)} = \frac{1}{N-1} \sum_{i=1}^{N-1} 2(p_{(i)} - q_{(i)}). \tag{1}$$

The value of $\tilde{G}(Y)$ is also provided by the ratio of the ''concentration area'' and $\frac{1}{2} \cdot \frac{N-1}{N}$. Multiplying both sides of (1) by $\frac{N-1}{N}$ gives

$$G(Y) = \frac{N-1}{N} \cdot \tilde{G}(Y) = \frac{1}{N} \sum_{i=1}^{N-1} 2(p_{(i)} - q_{(i)}) = \frac{1}{N} \sum_{i=1}^{N} G_{(i)}(Y). \tag{2}$$

In (2),

$$G_{(i)}(Y) = 2(p_{(i)} - q_{(i)}) \tag{3}$$

and $G_{(N)} = 0$. The point and synthetic Bonferroni (1930) measures are:

$$V_{(i)} = \frac{M(Y) - \bar{M}_{(i)}}{M(Y)}, \quad (i = 1, ..., N), \tag{4}$$

$$\tilde{V}(Y) = \frac{1}{N-1} \sum_{i=1}^{N-1} V_{(i)}(Y). \tag{5}$$

Multiplying both sides of (5) by $(N-1)/N$ gives:

$$V^{\star}(Y) = \frac{N-1}{N} \cdot \tilde{V}(Y) =$$

$$= \frac{1}{N} \sum_{i=1}^{N-1} V_{(i)}(Y) = \frac{1}{N} \sum_{i=1}^{N} V_{(i)}(Y) = \frac{1}{N} \sum_{i=1}^{N} \frac{M(Y) - \bar{M}_{(i)}}{M(Y)}. \tag{6}$$

De Vergottini (1940) showed that

$$V_{(i)}(Y) = \frac{M(Y) - \bar{M}_{(i)}}{M(Y)} = \frac{p_{(i)} - q_{(i)}}{p_{(i)}} = \rho_{(\frac{i}{N})}.$$

Thus:

$$G_{(i)}(Y) = 2(p_{(i)} - q_{(i)}) = \frac{M(Y) - \bar{M}_{(i)}}{M(Y)} \cdot 2p_{(i)}, \tag{7}$$

$$G(Y) = \frac{1}{N} \sum_{i=1}^{N} \frac{M(Y) - \bar{M}_{(i)}}{M(Y)} 2p_{(i)}. \tag{8}$$

Zenga (1984, 2007) introduced two new point and symthetic measures. In the present paper we shall consider the latter proposal that has been defined in the case of the frequency distribution framework. Thus, let $\{0 \le y_1 < ... < y_h < ... < y_r\}$ denote the set of the distinct values assumed by the variate Y over the k subpopulations and $\left\{n_{1.}, ..., n_{h.}, ..., n_{r.}; \sum_{h=1}^{r} n_{h.} = N\right\}$ be the corresponding frequencies. It is possible to report the whole $r \times k$ distribution as in Table 1, where n_{hg} denotes the frequency of the value y_h in the subpopulation g, $n_{h.} = \sum_{g=1}^{k} n_{hg}$, and $n_{.g} = \sum_{h=1}^{r} n_{hg}$.

TABLE 1. - *Bivariate $r \times k$ frequency distribution of the whole population partitioned into k subpopulations*

	Subpopulation					
	1	$\dots$	g	$\dots$	k	Total
y_1	n_{11}	$\dots$	n_{1g}	$\dots$	n_{1k}	$n_{1.}$
$\vdots$	$\vdots v$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	
y_h	n_{h1}	$\dots$	n_{hg}	$\dots$	n_{hk}	$n_{h.}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
y_r	n_{r1}	$\dots$	n_{rg}	$\dots$	n_{rk}	$n_{r.}$
Total	$n_{.1}$	$\dots$	$n_{.g}$	$\dots$	$n_{.k}$	N

Let us define, for the overall distribution $\{(y_h, n_{h.}) : h = 1, \dots, r\}$:

$$P_{h.} = P_{h.}(Y) = \textstyle\sum_{t=1}^{h} n_{t.} \qquad h = 1, \dots, r \tag{9}$$

$$S_{h.}(Y) = y_h \cdot n_{h.}, \qquad h = 1, \dots, r \tag{10}$$

$$Q_{h.} = Q_{h.}(Y) = \textstyle\sum_{t=1}^{h} S_{t.}(Y) = \sum_{t=1}^{h} y_t \cdot n_{t.}, \qquad h = 1, \dots, r \tag{11}$$

$$T = Q_{r.}(Y) = \textstyle\sum_{h=1}^{r} S_{h.}(Y) = \sum_{h=1}^{r} y_h \cdot n_{h.} \qquad . \tag{12}$$

At each y_h the whole population can split into two non overlapping groups: a lower group $\{(y_1, n_{1.}), ..., (y_h, n_{h.})\}$ including the first $P_{h.}$ units and the corresponding upper group $\{(y_{h+1}, n_{n+1.}), ..., (y_r, n_{r.})\}$ including the remaining $(N - P_{h.})$ units. Note that for $h = r$ the upper group is empty.

Let

$$\bar{M}_{h.}(Y) = \bar{M}_{h.} = \frac{Q_{h.}(Y)}{P_{h.}}, \qquad h = 1, \dots, r, \tag{13}$$

be the arithmetic mean (lower mean) of the lower group, and

$$M_{h.} = \overset{+}{M}_{h.}(Y) = \begin{cases} \frac{T(Y) - Q_{h.}(Y)}{N - P_{h.}}, & h = 1, \dots, r-1 \\ \overset{+}{M}_{r-1.}(Y) = y_r, & h = r \end{cases} \tag{14}$$

be the arithmetic mean (upper mean) of the upper group.

In order to measure the inequality between the lower group and the upper group, Zenga (2007) proposed the point index

$$I_h(Y) = \frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{h.}(Y)}{\overset{+}{M}_{h.}(Y)}, \qquad h = 1, \ldots, r \tag{15}$$

The synthetic Zenga's inequality measure $I(Y)$ is fournished by:

$$I(Y) = \sum_{h=1}^{r} I_h(Y) \cdot \frac{n_{h.}}{N} \ . \tag{15}$$

Now we point out that the Bonferroni point measure $V_{(i)}(Y) = \frac{M(Y) - \bar{M}_{(i)}}{M(Y)}$ and the "Gini" point index $G_{(i)}(Y) = \frac{M(Y) - \bar{M}_{(i)}}{M(Y)} \cdot 2p_{(i)}$ may not be constant for units taking the same value of Y. This behaviour of $V_{(i)}(Y)$ and $G_{(i)}(Y)$ is not reasonable in the decomposition by subpopulations because the $n_{h.}$ units with $Y = y_h$ may belong to different subpopulations. For the Bonferroni index, Zenga and Valli (2016) overcomed this situation by substituting each value $V_{(i)}(Y)$, corresponding to the $n_{h.}$units, with the value

$$V_{(P_{h.})} = \frac{M(Y) - \bar{M}_{h.}(Y)}{M(Y)} = V_h(Y), \qquad h = 1, ..., r. \tag{17}$$

Consequently these authors proposed as Bonferroni synthetic index $V(Y)$ the weighted mean of the r Bonferroni point indexes $V_h(Y)$ with weights $\frac{n_{h.}}{N}$:

$$V(Y) = \sum_{h=1}^{r} V_h(Y) \cdot \frac{n_{h.}}{N}. \tag{18}$$

For the Gini index, Zenga (2016b) proposed the substitution of each value $G_{(i)}(Y)$, corresponding to the $n_{h.}$ units, with their arithmetic mean

$$\begin{aligned} G_h(Y) &= M[G_i(Y) | Y = y_h] \\ &= \frac{M(Y) - \bar{M}_{h.}(Y)}{M(Y)} \cdot 2p_{h.} - \frac{n_{h.} - 1}{N} \cdot \frac{M(Y) - y_h}{M(Y)}, h = 1, ..., r, \end{aligned} \tag{19}$$

where $p_{h.} = \frac{P_{h.}}{N}$. Consequently, $G(Y)$ can be evaluated by

$$G(Y) = \sum_{h=1}^{r} \left[\frac{M(Y) - \bar{M}_{h.}(Y)}{M(Y)} \cdot 2p_{h.} - \frac{n_{h.} - 1}{N} \cdot \frac{M(Y) - y_h}{M(Y)} \right] \cdot \frac{n_{h.}}{N}. \tag{20}$$

3. DECOMPOSITION BY SUBPOPULATIONS

We need to introduce some additional definitions and notation. For the distribution $\{(y_h, n_{hg}) : h = 1, ..., r\}$ of the subpopulation g let:

$$P_{hg}(Y) = P_{hg} = \sum_{t=1}^{h} n_{tg}, \qquad h = 1, ...r, \tag{21}$$

is the relative frequency of subpopulation g in the upper group.

$$Q_{hg}(Y) = Q_{hg} = \sum_{t=1}^{h} y_t \cdot n_{tg} \qquad h = 1, ..., r, \tag{22}$$

$$T_g(Y) = Q_{rg}(Y), \tag{23}$$

$$M_g(Y) = T_g(Y)/n_{.g} \tag{24}$$

In addition, for the same subpopulation g, let

$$o(g) = \min h : n_{hg} > 0, \tag{25}$$

$$u(g) = \max h : n_{hg} > 0, \tag{26}$$

and define the lower mean $\bar{M}_{hg}(Y)$ and the upper mean $\overset{+}{M}_{hg}(Y)$ as follows:

$$\bar{M}_{hg} = \bar{M}_{hg}(Y) = \begin{cases} y_{o(g)} & \text{for } h < o(g) \\ Q_{hg}(Y)/P_{hg} & \text{for } h \geq o(g) \end{cases} \tag{27}$$

$$\overset{+}{M}_{hg} = \overset{+}{M}_{hg}(Y) = \begin{cases} \frac{T_g(Y) - Q_{hg}(Y)}{n_{.g} - P_{hg}} & \text{for } h < u(g) \\ y_{u(g)} & \text{for } h \geq u(g). \end{cases} \tag{28}$$

Note that $y_{o(g)}$ and $y_{u(g)}$ are, respectively, the minimum and the maximum value of Y in the subpopulation g.

The decomposition by subpopulations of the three indexes are obtained utilizing the ‘‘two-step’’ approach. In the first step the decompositions of the Zenga $I_h(Y)$, Bonferroni $V_h(Y)$ and Gini $G_h(Y)$ point indexes are obtained. In the second step, putting these decompositions in (16), (18) and (20), the decompositions of the corresponding synthetic measures are derived.

The lower mean $\bar{M}_{h.}(Y)$ is related to the k lower means $\bar{M}_{hl}(Y)$ by the relation

$$\bar{M}_{h.}(Y) = \sum_{l=1}^{k} \bar{M}_{hl}(Y) \cdot p(l|h), \tag{29}$$

where

$$p(l|h) = \frac{P_{hl}}{P_{h.}}, \qquad h = 1, ..., r; l = 1, ..., k \tag{30}$$

is the relative frequency of subpopulation l in the lower group $\{Y \leq y_h\}$. Note that $\sum_{l=1}^{k} p(l|h) = 1$. Analogously,

$$\overset{+}{M}_{h.}(Y) = \sum_{g=1}^{k} \overset{+}{M}_{hg}(Y) \cdot a(g|h), \tag{31}$$

where

$$a(g|h) = \begin{cases} \frac{n_{.g} - P_{hg}}{N - P_{h.}} & \text{for } h = 1, \dots, r-1 \\ \frac{n_{rg}}{n_{r.}} & \text{for } h = r \end{cases} \tag{32}$$

is the relative frequency of subpopulation g in the upper group.

Putting (29) and (31) in the numerator of (15), the following $k \times k$ additive decomposition of $I_h(Y)$ is obtained:

$$I_h(Y) = \sum_{l=1}^{k}\sum_{g=1}^{k} \frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} p(l|h)a(g|h) = \sum_{l=1}^{k}\sum_{g=1}^{k} B_{hlg}(Y). \tag{33}$$

It is worth to remark that, starting from the $k \times k$ decomposition (33), the following k additive decomposition of $I_h(Y)$ is obtained:

$$I_h(Y) = \sum_{l=1}^{k} p(l|h) \left\{ \sum_{g=1}^{k} \frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} a(g|h) \right\} =$$

$$= \sum_{l=1}^{k} \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l|h) = \sum_{l=1}^{k} B_{hl.}(Y), \tag{34}$$

where,

$$B_{hl.}(Y) = \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l|h). \tag{35}$$

Note that $B_{hl.}(Y)$ is the product of the relative variation of the lower mean $\bar{M}_{hl}(Y)$ w.r.t. the upper mean $\overset{+}{M}_{h.}(Y)$ and the relative frequency $p(l|h)$. In other words, formula (34) shows that the point index $I_h(Y)$ is the weighted mean of the k relative variations $\left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right]$ with weights $p(l|h)$. Thus, $B_{hl.}(Y)$ can be interpreted as the contribution of the subpopulation l to the point inequality index $I_h(Y)$. In addition, $B_{hl.}(Y)$ can be split into the following within $B_{hlW}(Y)$ and between $B_{hlB}(Y)$ components:

$$B_{hl.}(Y) = \sum_{g=1}^{k} B_{hlg}(Y) = B_{hll}(Y) + \sum_{g \neq l} B_{hlg}(Y) = B_{hlW}(Y) + B_{hlB}(Y). \tag{36}$$

Consequently, the within and the between component of the point index $I_h(Y)$ are given by:

$$I_h(Y) = \sum_{l=1}^{k} B_{hl.}(Y) = \sum_{l=1}^{k} B_{hlW}(Y) + \sum_{l=1}^{k} B_{hlB}(Y) = B_{h.W}(Y) + B_{h.B}(Y). \tag{37}$$

In (37) $B_{h.W}(Y) = \sum_{l=1}^{k} B_{hlW}(Y)$ can be interpreted as the within contribution of all the k subpopulations to $I_h(Y)$, and $B_{h.B}(Y) = \sum_{l=1}^{k} B_{hlB}(Y)$ can be interpreted as the between contrubution of all the k subpopulations to $I_h(Y)$.

Finally, second step, putting the decompositions (33), (34) and (37) of $I_h(Y)$ in (16), gives the corresponding decompositions of the synthetic index $I(Y)$. Thus, using (33) in (16) gives

$$I(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \left\{ \sum_{h=1}^{r} B_{hlg}(Y) \cdot \frac{n_{h.}}{N} \right\} = \sum_{l=1}^{k} \sum_{g=1}^{k} B_{.lg}(Y). \tag{38}$$

In (38) $B_{.lg}(Y)$ is the weighted mean of the r contributions $B_{hlg}(Y)$ with weights $\frac{n_{h.}}{N}$. Now, using (34) in (16) gives:

$$I(Y) = \sum_{h=1}^{r} \left\{ \sum_{l=1}^{k} B_{hl.}(Y) \right\} \frac{n_{h.}}{N} = \sum_{l=1}^{k} \sum_{h=1}^{r} B_{hl.}(Y) \frac{n_{h.}}{N} = \sum_{l=1}^{k} B_{.l.}(Y), \tag{39}$$

where

$$B_{.l.}(Y) = \sum_{h=1}^{r} B_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{g=1}^{k} B_{.lg}(Y) \tag{40}$$

is the contribution of subpopulation l to $I(Y)$. Finally, using (37) in (16) gives:

$$I(Y) = \sum_{l=1}^{k} \sum_{h=1}^{r} \{B_{hlW}(Y) + B_{hlB}(Y)\} \frac{n_{h.}}{N} = \sum_{l=1}^{k} \{B_{.lW}(Y) + B_{.lB}(Y)\}, \tag{41}$$

and

$$I(Y) = B_{..W}(Y) + B_{..B}(Y). \tag{42}$$

In (41) $B_{.lW} = \sum_{h=1}^{r} B_{hll} \cdot \frac{n_{h.}}{N}$ and $B_{.lB} = \sum_{(g:g\neq l)} \left\{ \sum_{h=1}^{r} B_{hlg} \cdot \frac{n_{h.}}{N} \right\}$ are the within and between parts of $B_{.l.}(Y)$, respectively. In (42)

$$B_{..W}(Y) = \sum_{l=1}^{k} \left\{ \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hl}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(l|h) \cdot \frac{n_{h.}}{N} \right\}, \tag{43}$$

and

$$B_{..B}(Y) = \sum_{l=1}^{k} \sum_{(g:g\neq l)} \left\{ \sum_{h=1}^{r} \left[\frac{\overset{+}{M}_{hg}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] p(l|h) \cdot a(g|h) \cdot \frac{n_{h.}}{N} \right\} \tag{44}$$

are the within and the between part of the synthetic index $I(Y)$, respectively. Note that in $B_{..W}(Y)$ only comparisons between upper and lower means of the same subpopulations are involved, while in $B_{..B}(Y)$ only comparisons between upper and lower means of different subpopulations are involved.

For the illustration of the decomposition by subpopulations of the Bonferroni point index, we need to introduce the following popular relation

$$M(Y) = \sum_{g=1}^{k} M_g(Y) \cdot \frac{n_{.g}}{N}. \tag{45}$$

Using the relations (45) and (29) in the numerator of (17), Zenga and Valli (2016) obtained, after some steps, the following decompositions:

$$V_h(Y) = \sum_{l=1}^{k}\sum_{g=1}^{k} \frac{M_g(Y) - \bar{M}_{hl}(Y)}{M(Y)} p(l|h) \frac{n_{.g}}{N} = \sum_{l=1}^{k}\sum_{g=1}^{k} V_{hlg}(Y); \tag{46}$$

$$V_h(Y) = \sum_{l=1}^{k} \frac{M(Y) - \bar{M}_{hl}(Y)}{M(Y)} p(l|h) = \sum_{l=1}^{k} V_{hl.}(Y); \tag{47}$$

$$V_{hl.}(Y) = V_{hlW}(Y) + V_{hlB}(Y), \tag{48}$$

where

$$V_{hlW}(Y) = \frac{M_l(Y) - \bar{M}_{hl}(Y)}{M(Y)} p(l|h) \frac{n_{.l}}{N} \tag{49}$$

and

$$V_{hlB}(Y) = \sum_{(g \neq l)} \frac{M_g(Y) - \bar{M}_{hl}(Y)}{M(Y)} p(l|h) \frac{n_{.g}}{N}; \tag{50}$$

$$V_h(Y) = \sum_{l=1}^{k} \{V_{hlW}(Y) + V_{hlB}(Y)\} = V_{h.W}(Y) + V_{h.B}(Y). \tag{51}$$

Using, second step, the decompositions (46), (47) and (51) in (18), the corresponding decompositions of the Bonferroni synthetic index $V(Y)$, are obtained. Thus:

$$V(Y) = \sum_{h=1}^{r} V_h(Y) \cdot \frac{n_{h.}}{N} = \sum_{l=1}^{k}\sum_{g=1}^{k} V_{.lg}(Y), \tag{52}$$

where

$$V_{.lg}(Y) = \sum_{h=1}^{r} V_{hlg}(Y) \cdot \frac{n_{h.}}{N}; \tag{53}$$

$$V(Y) = \sum_{h=1}^{r}\sum_{l=1}^{k} V_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{l=1}^{k}\sum_{h=1}^{r} V_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{l=1}^{k} V_{.l.}(Y), \tag{54}$$

where

$$V_{.l.}(Y) = \sum_{h=1}^{r} V_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{g=1}^{k} V_{.lg}(Y) \tag{55}$$

is the contribution of subpopulation l to $V(Y)$. In addition utilizing (48) in (55) gives

$$V_{.l.}(Y) = \sum_{h=1}^{r} [V_{hlW}(Y) + V_{hlB}(Y)] \cdot \frac{n_{h.}}{N} = V_{.lW}(Y) + V_{.lB}(Y). \tag{56}$$

In (56), $V_{.lW}(Y) = \sum_{h=1}^{r} V_{hlW}(Y) \cdot \frac{n_{h.}}{N}$ and $V_{.lB}(Y) = \sum_{h=1}^{r} V_{hlB}(Y) \cdot \frac{n_{h.}}{N}$, are the within and the between part of $V_{.l.}(Y)$, respectively. Now, putting (56) in (54) gives

$$V(Y) = \sum_{l=1}^{k} V_{.l.}(Y) = \sum_{l=1}^{k} \{V_{.lW}(Y) + V_{.lB}(Y)\} = V_{..W}(Y) + V_{..B}(Y). \tag{57}$$

In (57), $V_{..W}(Y) = \sum_{l=1}^{k} V_{.lW}(Y)$ and $V_{..B}(Y) = \sum_{l=1}^{k} V_{.lB}(Y)$ are the within and the between components of the Bonferroni synthetic index $V(Y)$.

For the illustration of the decomposition by subpopulations of the Gini point index, we need to introduce the following relations:

$$y_h = \sum_{l=1}^{k} y_h \cdot f(l|h), \tag{58}$$

where

$$f(l|h) = \frac{n_{hl}}{n_{h.}}, \qquad h = 1, ..., r; \quad l = 1, 2, ..., k. \tag{59}$$

is the relative frequency of subpopulation l in the group of the $n_{h.}$ units with $Y = y_h$. Note that $\sum_{l=1}^{k} f(l|h) = 1$. Using the relations (45) and (58) in the difference $(M(Y) - y_h)$ gives the following decomposition

$$(M(Y) - y_h) = \sum_{l=1}^{k} \sum_{g=1}^{k} (M_g(Y) - y_h) f(l|h) \frac{n_{.g}}{N}. \tag{60}$$

Then, using the decomposition (46) of the Bonferroni point index and the relation (60) in formula (19), Zenga (2016b) obtained, after some steps, the following decompositions of the Gini point measure:

$$G_h(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} C_{hlg}(Y), \tag{61}$$

in (61):

$$C_{hlg}(Y) = \frac{M_g - \bar{M}_{hl}}{M(Y)} p(l|h) \frac{n_{.g}}{N} 2p_{h.} - \frac{M_g - y_h}{M(Y)} f(l|h) \frac{n_{.g}}{N} \frac{n_h - 1}{N}; \tag{62}$$

$$G_h(Y) = \sum_{l=1}^{k} \left\{ \sum_{g=1}^{k} C_{hlg}(Y) \right\} = \sum_{l=1}^{k} C_{hl.}(Y), \tag{63}$$

where

$$C_{hl.}(Y) = \frac{M(Y) - \bar{M}_{hl}(Y)}{M(Y)} p(l|h) 2p_{h.} - \frac{M(Y) - y_h}{M(Y)} f(l|h) \frac{n_h - 1}{N} \tag{64}$$

is the contribution of subpopulation l to $G_h(Y)$;

$$C_{hl.}(Y) = C_{hlW}(Y) + C_{hlB}(Y), \tag{65}$$

where

$$C_{hlW} = \frac{M_l - \bar{M}_{hl}}{M(Y)} p(l|h) \frac{n_{.l}}{N} 2p_{h.} - \frac{M_l - y_h}{M(Y)} f(l|h) \frac{n_{.l}}{N} \frac{n_h - 1}{N}, \tag{66}$$

and

$$C_{hlB} = \sum_{g \neq l} \left\{ \frac{M_g - \bar{M}_{hl}}{M(Y)} p(l|h) \frac{n_{.g}}{N} 2p_{h.} - \frac{M_g - y_h}{M(Y)} f(l|h) \frac{n_{.g}}{N} \frac{n_h - 1}{N} \right\}; \tag{67}$$

$$G_h(Y) = \sum_{l=1}^{K} \{C_{hlW}(Y) + C_{hlB}(Y)\} = C_{h.W}(Y) + C_{h.B}(Y), \tag{68}$$

in (68): $C_{h.W}(Y)$ and $C_{h.B}(Y)$ are the within and the between part of $G_h(Y)$ respectively.

Using, second step, the decompositions (61), (63), (65) and (68) in (20), the corresponding decompositions of the Gini synthetic index $G(Y)$, are obtained. Thus:

$$G(Y) = \sum_{l=1}^{k} \sum_{g=1}^{k} \left\{ \sum_{h=1}^{r} C_{hlg}(Y) \cdot \frac{n_{h.}}{N} \right\} = \sum_{l=1}^{k} \sum_{g=1}^{k} C_{.lg}(Y), \tag{69}$$

in (69) $C_{.lg}(Y)$ is the weighted mean of $C_{hlg}(Y)$ with weights $\frac{n_{h.}}{N}$;

$$G(Y) = \sum_{h=1}^{r} \sum_{l=1}^{k} C_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{l=1}^{k} \sum_{h=1}^{r} C_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{l=1}^{k} C_{.l.}(Y), \tag{70}$$

where

$$C_{.l.}(Y) = \sum_{h=1}^{r} C_{hl.}(Y) \cdot \frac{n_{h.}}{N} = \sum_{g=1}^{k} C_{.lg}(Y) \tag{71}$$

is the contribution of subpopulation l to $G(Y)$;

$$C_{.l.}(Y) = \sum_{g=1}^{k} C_{.lg}(Y) = C_{.ll}(Y) + \sum_{g \neq l} C_{.lg}(Y) = C_{.lW}(Y) + C_{.lB}(Y), \tag{72}$$

in (72), $C_{.lW}(Y)$ and $C_{.lB}(Y)$ are the within and the between part of $C_{.l.}$, respectively;

$$G(Y) = \sum_{l=1}^{k} C_{.l.}(Y) = \sum_{l=1}^{k} \left\{ C_{.ll}(Y) + \sum_{g \neq l} C_{.lg}(Y) \right\} = C_{..W}(Y) + C_{..B}(Y), \tag{73}$$

where

$$C_{..W} = \sum_{l=1}^{k} \sum_{h=1}^{r} \left(\frac{M_l - \bar{M}_{hl}}{M(Y)} p(l|h) \frac{n_{.l}}{N} 2p_{h.} - \frac{M_l - y_h}{M(Y)} \frac{n_{.l}}{N} f(l|h) \frac{n_{h.} - 1}{N} \right) \frac{n_{h.}}{N} \tag{74}$$

and

$$C_{..B} = \sum_{l=1}^{k} \sum_{g \neq l} \sum_{h=1}^{r} \frac{n_{.g}}{N} \left(\frac{M_g - \bar{M}_{hl}}{M(Y)} p(l|h) 2p_{h.} - \frac{M_g - y_h}{M(Y)} f(l|h) \frac{n_{h.} - 1}{N} \right) \frac{n_{h.}}{N} \tag{75}$$

are the within and between part of $G(Y)$, respectively.

Now we have to remark that, the decompositions illustrated in this section can be utilized in the case of non-overlapping subpopulations as well as in the overlapping case.

4. NUMERICAL ILLUSTRATIONS OF THE DECOMPOSITIONS BY SUBPOPULATIONS OF THE THREE INEQUALITY INDEXES

It is possible to report in a $r \times k$ bivariate table some useful results obtained in Section 3. We have reported in each cell hl, $(h = 1, ..., r; l = 1, ...k)$, of Table 12 the product $\frac{n_{h.}}{N} B_{hl.}(Y)$. According to the relation (39) the sum of these $r \times k$ values $\sum_{h=1}^{r} \sum_{l=1}^{k} \frac{n_{h.}}{N} B_{hl.}(Y)$ gives the value of the synthetic Zenga index $I(Y)$. The sum of the values $\sum_{l=1}^{k} \frac{n_{h.}}{N} B_{hl.}(Y)$ reported in the k cells of row h gives $\frac{n_{h.}}{N} \cdot I_h(Y)$. These r products $\frac{n_{h.}}{N} \cdot I_h(Y)\,, h = 1, ..., r$, are reported in the "marginal" column of the bivariate Table, and their sum gives $I(Y)$. Moreover, the sum of the values reported in the r cells of column l gives $\sum_{h=1}^{r} \frac{n_{h.}}{N} B_{hl.}(Y) = B_{.l.}(Y)$. These latter k values $B_{.l.}(Y)$ can be reported in the "marginal" row. Obviously, the sum $\sum_{l=1}^{k} B_{.l.}(Y)$ is equal to $I(Y)$.

The value of the synthetic Zenga index, $I(Y) = \sum_{h=1}^{r} I_h(Y) \cdot \frac{n_{h.}}{N}$, is equal to the sum of the areas of r rectangles with bases $\frac{n_{h.}}{N}$ and heights $I_h(Y)$, $h = 1, ..., r$. To draw the inequality diagram of $I_h(Y)$, it is necessary, first of all, to obtain the r points of coordinates $(p_{h.}, I_h(Y))$. Then, we obtain the r rectangles by the following procedure: the first rectangle has abscissas in the interval $[0, p_{1.}]$ and ordinates in the interval $[0, I_1(Y)]$. The h^{th} rectangle, $h = 2, ..., r$, has abscissas in the interval $[p_{h-1.}, p_{h.}]$ and ordinates in the interval $[0, I_h(Y)]$. Now, from the relation $\frac{n_{h.}}{N} \cdot I_h(Y) = \sum_{l=1}^{k} \frac{n_{h.}}{N} \cdot B_{hl.}$, it derives that the area of the h^{th} rectangle can be split in the areas of l sub-rectangles with (common) base $\frac{n_{h.}}{N}$ and heights

$B_{hl.}, (l = 1, ..., k)$. Consequently, $\frac{n_{h.}}{N} \cdot B_{hl.}$ can be interpreted as the contribution of subpolulation l to the contribution $\frac{n_{h.}}{N} \cdot I_h(Y)$.

Obviously, this kind of representations can be extended to the Bonferroni and to the Gini indexes, too.

The decompositions of the three indexes are illustrated by a numerical example of a population of $N = 20$ units partitioned into $k = 5$ non overlapping subpopulations (Section 4.1), and by a numerical example of a population of $N = 7$ units partitioned into $k = 3$ overlapping subpopulations (Section 4.2).

4.1 *Numerical example of a population of $N = 20$ units partitioned into $K = 5$ non overlapping subpopulations*

Table 3 reports a 20×5 bivariate distribution of a population of 20 units partitioned into $k = 5$ non overlapping subpopulations. The variate Y assumes the following $r = 20$ distinct values: $(y_1 = 0; y_2 = 1; ...; y_{19} = 18; y_{20} = 29)$. Table 4 shows the calculation for the three point indexes: $I_h(Y)$, $V_h(Y)$ and $G_h(Y)$. Tables 5, 6 and 7 report, for each subpopulation, the lower and upper means. Tables 8 and 9 report the relative frequencies $p(l|h)$ and $a(g|h)$. Finally, Tables 10, 11 and 12 report for each synthetic index $(G(Y), V(Y)\text{andI(Y)})$ the joint contributions $(\frac{1}{20} \cdot C_{hl.}, \frac{1}{20} \cdot V_{hl.}, \frac{1}{20} \cdot B_{hl.})$, and the corresponding row and column marginal contributions. It seems useful to give more informations on these Tables. In the example at hand the supports of the five non overlapping subpopulations are:

$$[y_1 = 0; ...; y_4 = 3],\ [y_5 = 4; ...; y_8 = 7],\ [y_9 = 8; ...; y_{12} = 11],$$
$$[y_{13} = 12; ...; y_{16} = 15],\ [y_{17} = 16; ...; y_{19} = 18;\ y_{20} = 29].$$

Consequently, see Table 3, the cumulative frequency P_{hl} are equal to zero for: $[l = 2 \text{ and } 1 \leq \text{h} \leq 4]$, $[l = 3 \text{ and } 1 \leq \text{h} \leq 8]$, $[l = 4 \text{ and } 1 \leq \text{h} \leq 12]$, $[l = 5 \text{ and } 1 \leq \text{h} \leq 16]$. This explains why the "corresponding" relative frequencies $p(l|h) = (P_{hl}/P_{h.}) = (P_{hl}/h)$, reported in Table 8, are equal to zero. Now we report the detail for the calculation of the contributions $\frac{1}{20} B_{6l.} = \frac{1}{20} \left\{ (\overset{+}{M}_{6.} - \bar{M}_{6l})/\overset{+}{M}_{6.} \right\} p(l/h)$ reported in the 6^{th} row of Table 12.

$$\frac{1}{20} B_{61.} = \frac{1}{20} \left\{ (\overset{+}{M}_{6.} - \bar{M}_{61})/\overset{+}{M}_{6.} \right\} p(1|6) = 0.05\{(13.214 - 1.5)/13.214\}0.666 =$$
$$= 0.05 \cdot 0.88648 \cdot 0.666 = 0.0295.$$

$$\frac{1}{20} B_{62.} = \frac{1}{20} \left\{ (\overset{+}{M}_{6.} - \bar{M}_{62})/\overset{+}{M}_{6.} \right\} p(2|6) = 0.05\{(13.214 - 4.5)/13.214\}0.333 =$$
$$= 0.05 \cdot 0.65945 \cdot 0.333 = 0.01099.$$

Now, $p(3|6) = p(4|6) = p(5|6) = 0$, and consequently $0.05 \cdot B_{63.} = 0.05 \cdot B_{64.} = = 0.05 \cdot B_{65.} = 0$.

Interesting informations regarding the decomposition by 5 non overlapping subpopulations are reported in Tables 10, 11 and 12.

a) Table 12. The value $\frac{1}{20} \cdot I_6 = 0.04054$ reported in the marginal column is the contribution to the synthetic index $I(Y) = 0.74506$ related to the 6^{th} interval $(0.25 < p_{h.} \leq 0.30)$. This value is also equal to the area of the 6^{th} rectangle. The value $\frac{1}{20} \cdot B_{61.} = 0.02955$ is the contribution of the subpopulation $l = 1$ related to the (same) interval. The value $B_{.1.} = \sum_{h=1}^{20} \frac{1}{20} B_{h1.} = 0.4614$ is the simple mean of the $r = 20$ contributions $B_{h1.}$, and it is the contribution of the subpopulation $l = 1$ to the synthetic index $I(Y) = 0,74506$. In the graph of $I_h(Y)$, $B_{.1.}$ is the sum of the $N = 20$ sub-rectangles related to the subpopulation $l = 1$.
b) Tables 10 and 11. It is worth to remark that in the case of the Bonferroni and Gini indexes there are negative contributions regarding the subpopulations $l = 4$ and $l = 5$. Let us consider for the Bonferroni index the calculation of the the contribution $\frac{1}{20} \cdot V_{15\,4.} = -0.003$:

$$\frac{1}{20} \cdot V_{15\,4.} = 0.05 \cdot [(M(Y) - \bar{M}_{15\,4})/M(Y)] \cdot p(4|15) =$$
$$= 0.05 \cdot [(10 - 13)/10] \cdot 0.2 = 0.05 \cdot [-0.3] \cdot 0.2 = -0.003.$$

Note that in the example at hand, $n_{h.} = 1$ for $h = 1, .., 20$, and the Gini contributions $C_{lh.}$ are given by $C_{hl.} = [(M(Y) - \bar{M}_{hl})/M(Y)] \cdot p(l/h) \cdot 2p_{h.} = V_{hl.} \cdot 2p_{h.}$. Let us consider now the calculation of the contribution $\frac{1}{20} \cdot C_{15\,4.} = -0.0045$: $\frac{1}{20} \cdot C_{15\,4.} = \ = (-0.003) \cdot 2 \cdot (15/20) = -0.0045$. In conclusion if the lower mean $-M_{hl} > M(Y)$ we have negative contributions for the Bonferroni and the Gini indexes. Viceversa, in the case of the Zenga index $\overset{+}{M}_{h.} > \bar{M}_{hl}$ and, consequently, the contributions $\frac{1}{20} B_{hl.}(Y) = \frac{1}{20} \cdot \left[\frac{\overset{+}{M}_{h.}(Y) - \bar{M}_{hl}(Y)}{\overset{+}{M}_{h.}(Y)} \right] \cdot p(l|h)$ cannot assume negative values.
c) The Tables 10, 11, and 12 do not report the within and the between parts of the contributions $B_{hl.}$, $V_{hl.}$ and $C_{hl.}$. We show now the calculation of the within and the between parts of the Bonferroni contribution $V_{hl.} = V_{hlW} + V_{hlB}$, for $h = 15$ and $l = 4$. $V_{15\,4W} = [(M_4 - \bar{M}_{15\,4})/M(Y)] \cdot p(4|15) \cdot \frac{n_{.4}}{N}$, and $V_{15\,4B} = \sum_{g \neq 4} \left[(M_g - \bar{M}_{15\,4})/M(Y) \right] \cdot p(4|15) \cdot \frac{n_{.g}}{N}$. Now: $(n_{.g}/20) = 0.2$ for $g = 1, ..., 5$ and $p(4|15) = 0.2$, thus $p(4|15) \cdot (n_{.g}/20)$=0.04 ,for $g = 1, ..., 5$; $M(Y) = 10$, $\bar{M}_{15\,4} = 13$; the means of the five subpopulations are $M_1 = 1.5$, $M_2 = 5.5$, $M_3 = 9.5$, $M_4 = 13.5$ and $M_5 = 20$.

We are ready to report in the following Table 2 the calculation of the five contributions $V_{15\,4g}$.

Thus:
$V_{15\,4W} = V_{15\,44} = +0.002$;
$V_{15\,4B} = (V_{15\,41} + V_{15\,42} + V_{15\,43} + V_{15\,45}) = (-0.46 - 0.30 - 0.014 + 0.028) = -0.062$
$V_{15\,4.} = +0.002 - 0.062 = -0.06$.

Note that the within part is never negative because $(M_l - \bar{M}_{hl}) \geq 0$, while the between part can assume negative as well as positive values. Viceversa, in the case of the Zenga index the differences $(\overset{+}{M}_{hl} - \bar{M}_{hl})$ and $(\overset{+}{M}_{hg} - \bar{M}_{hl})$, $(g \neq l)$, are never negatives. Consequently, the corresponding between parts are never negative.

In applied analysis it may be useful to aggregate the N units in groups delimitated by quartiles, quintiles, ... Using the quintiles the five groups of the numerical example at hand are: $(y_{1+4\cdot(s-1)}, ..., y_{4\cdot s})$, $s = 1, \ldots, 5$. These groups are the k subpopulations of the present example. In the case of the $r \times k$ bivariate Table 11, each h^{th} row reports the five contributions $\frac{n_{h.}}{20} \cdot V_{hl.}$ and the marginal contribution $\frac{n_{h.}}{20} \cdot V_{h..} = \frac{n_{h.}}{20} \cdot V_{h}$. These six contributions are related to the interval $D_h = (p_{h-1.} \dashv p_{h.})$ of the Bonferroni graph. In the case of the ''quintiles'' we need to aggregate the first 4 rows $(h = 1, ..., 4)$, the second 4 rows $(h = 5, ..., 8)$,etc. . In this way we obtain contributions that are related to the intervals: $D_{(1)} = (0 \dashv 0.2)$, $D_{(2)} = (0.2 \dashv 0.4)$, ..., $D_{(5)} = (0.8 \dashv 1.0)$.

TABLE 2. - *Calculation of the quantities needed for the evaluation of* $V_{15\,4g}$

g	M_g	$\bar{M}_{15\,4}$	$M(Y)$	$(M_g - 13)/10$	$p(4\|15) \cdot (n_{.g}/20)$	$V_{15\,4g}$
1	1.5	13	10	-1.15	0.04	-0.046
2	5.5	13	10	-0.75	0.04	-0.030
3	9.5	13	10	-0.35	0.04	-0.014
4	13.5	13	10	+0.05	0.04	+0.002
5	20	13	10	+0.7	0.04	+0.028
Total						-0.06 $= V_{15\,4.}$

Table 14 reports the decompositions of the Bonferroni index $V(Y)$ according to the five intervals $D_{(s)}$ – related to the cumulative frequency $p_{h.}$ - -and the five subpopulations. In each cell sl of Table 14 we report: the contributions

$$V_{(s)l.} = \sum_{h=1+4(s-1)}^{4s} \frac{1}{20} V_{hl.}$$

and the corresponding within $V_{(s)lW}$ and between $V_{(s)lB}$ parts, where:

$$V_{(s)lW} = \sum_{h=1+4(s-1)}^{4s} \frac{1}{20} V_{hlW} \quad \text{and} \quad V_{(s)lB} = \sum_{h=1+4(s-1)}^{4s} \frac{1}{20} V_{hlB}.$$

In each cell s of the marginal column we report: the contribution $V_{(s)..} = \sum_{l=1}^{5} V_{(s)l.}$ and the corresponding within $V_{(s).W} = \sum_{l=1}^{5} V_{(s)lW}$ and between $V_{(s).B} = \sum_{l=1}^{5} V_{(s)lB}$ parts. Obviously $\sum_{s=1}^{5} V_{(s)..} = V(Y)$, and, $\sum_{s=1}^{5} V_{(s).W} = V_{..W}$ and $\sum_{s=1}^{5} V_{(s).B} = V_{..B}$ are respectively the within and the between parts of $V(Y)$. In each cell l of the marginal row we report the contribution $V_{.l.} = \sum_{s=1}^{5} V_{(s)l.}$ and the corresponding within $V_{.lW} = \sum_{s=1}^{5} V_{(s)lW}$ and between $V_{.lB} = \sum_{s=1}^{5} V_{(s)lB}$ parts. Obviously $V_{..W} = \sum_{l=1}^{5} V_{.lW}$ and $V_{..B} = \sum_{l=1}^{5} V_{.lB}.$

TABLE 3. - *Joint frequencies n_{hl}, values of $o(l)$ and $u(l)$, total frequencies $n_{.l}$ and $n_{h.}$, cumulative frequencies P_{hl} of a population of $N = 20$ units partitioned into $k = 5$ non overlapping subpopulations*

n_{hl}		Subpopulations l						Cumulative frequencies P_{hl}				
h	y_h	1	2	3	4	5	$n_{h.}$	P_{h1}	P_{h2}	P_{h3}	P_{h4}	P_{h5}
1	0	1	0	0	0	0	1	1	0	0	0	0
2	1	1	0	0	0	0	1	2	0	0	0	0
3	2	1	0	0	0	0	1	3	0	0	0	0
4	3	1	0	0	0	0	1	4	0	0	0	0
5	4	0	1	0	0	0	1	4	1	0	0	0
6	5	0	1	0	0	0	1	4	2	0	0	0
7	6	0	1	0	0	0	1	4	3	0	0	0
8	7	0	1	0	0	0	1	4	4	0	0	0
9	8	0	0	1	0	0	1	4	4	1	0	0
10	9	0	0	1	0	0	1	4	4	2	0	0
11	10	0	0	1	0	0	1	4	4	3	0	0
12	11	0	0	1	0	0	1	4	4	4	0	0
13	12	0	0	0	1	0	1	4	4	4	1	0
14	13	0	0	0	1	0	1	4	4	4	2	0
15	14	0	0	0	1	0	1	4	4	4	3	0
16	15	0	0	0	1	0	1	4	4	4	4	0
17	16	0	0	0	0	1	1	4	4	4	4	1
18	17	0	0	0	0	1	1	4	4	4	4	2
19	18	0	0	0	0	1	1	4	4	4	4	3
20	29	0	0	0	0	1	1	4	4	4	4	4
$n_{.l}$		4	4	4	4	4	20					

$o(l)$	1	5	9	13	17
$u(l)$	4	8	12	16	20

TABLE 4. - *Calculation of $I_h(Y)$, $V_h(Y)$ and $G_h(Y)$ point inequality measures*

h	$Q_{h.}$	$T - Q_{h.}$	$P_{h.}$	$\bar{M}_{h.}$	$\overset{+}{M}_{h.}$	I_h	V_h	$2p_{h.}$	G_h
1	0	200	1	0.0	10.526	1.000	1.00	0.1	0.10
2	1	199	2	0.5	11.055	0.955	0.95	0.2	0.19
3	3	197	3	1.0	11.588	0.914	0.90	0.3	0.27
4	6	194	4	1.5	12.125	0.8763	0.85	0.4	0.34
5	10	190	5	2.0	12.166	0.842	0.80	0.5	0.40
6	15	185	6	2.5	13.214	0.811	0.75	0.6	0.45
7	21	179	7	3.0	13.769	0.782	0.70	0.7	0.49
8	28	172	8	3.5	14.333	0.756	0.65	0.8	0.52
9	36	164	9	4.0	14.909	0.732	0.60	0.9	0.54
10	45	155	10	4.5	15.500	0.710	0.55	1.0	0.55
11	55	145	11	5.0	16.111	0.690	0.50	1.1	0.55
12	66	134	12	5.5	16.750	0.672	0.45	1.2	0.54
13	78	122	13	6.0	17.428	0.656	0.40	1.3	0.52
14	91	109	14	6.5	18.166	0.642	0.35	1.4	0.49
15	105	95	15	7.0	19.000	0.632	0.30	1.5	0.45
16	120	80	16	7.5	20.000	0.625	0.25	1.6	0.40
17	136	64	17	8.0	21.333	0.625	0.20	1.7	0.34
18	153	47	18	8.5	23.500	0.638	0.15	1.8	0.27
19	171	29	19	9.0	29.000	0.690	0.10	1.9	0.19
20	200	0.00	20	10.0	29.000	0.655	0.00	2.0	0.00
Total						14.901	10.45		7.6
						$I(Y) =$	$V(Y) =$		$G(Y) =$
						0.74506	0.5225		0.38

TABLE 5. - *Cumulative incomes $Q_{hl}(Y)$ and lower means $\bar{M}_{hl}(Y)$*

		Cumulative incomes					Lower means				
h	y_h	Q_{h1}	Q_{h2}	Q_{h3}	Q_{h4}	Q_{h5}	$\bar{M}_{h1}$	$\bar{M}_{h2}$	$\bar{M}_{h3}$	$\bar{M}_{h4}$	$\bar{M}_{h5}$
1	0	0	0	0	0	0	0.0	4.0	8.0	12.0	16.0
2	1	1	0	0	0	0	0.5	4.0	8.0	12.0	16.0
3	2	3	0	0	0	0	1.0	4.0	8.0	12.0	16.0
4	3	6	0	0	0	0	1.5	4.0	8.0	12.0	16.0
5	4	6	4	0	0	0	1.5	4.0	8.0	12.0	16.0
6	5	6	9	0	0	0	1.5	4.5	8.0	12.0	16.0
7	6	6	15	0	0	0	1.5	5.0	8.0	12.0	16.0
8	7	6	22	0	0	0	1.5	5.5	8.0	12.0	16.0
9	8	6	22	8	0	0	1.5	5.5	8.0	12.0	16.0
10	9	6	22	17	0	0	1.5	5.5	8.5	12.0	16.0
11	10	6	22	27	0	0	1.5	5.5	9.0	12.0	16.0
12	11	6	22	38	0	0	1.5	5.5	9.5	12.0	16.0
13	12	6	22	38	12	0	1.5	5.5	9.5	12.0	16.0
14	13	6	22	38	25	0	1.5	5.5	9.5	12.5	16.0
15	14	6	22	38	39	0	1.5	5.5	9.5	13.0	16.0
16	15	6	22	38	54	0	1.5	5.5	9.5	13.5	16.0
17	16	6	22	38	54	16	1.5	5.5	9.5	13.5	16.0
18	17	6	22	38	54	33	1.5	5.5	9.5	13.5	16.5
19	18	6	22	38	54	51	1.5	5.5	9.5	13.5	17.0
20	29	6	22	38	54	80	1.5	5.5	9.5	13.5	20.0

TABLE 6. - *Upper group frequencies* $(n_{.l} - P_{hl})$ *and upper group incomes* $(T_l - Q_{hl})$

		Upper group frequencies					Upper group incomes				
h	y_h	4-P_{h1}	4-P_{h2}	4-P_{h3}	4-P_{h4}	4-P_{h5}	6-Q_{h1}	22-Q_{h2}	38-Q_{h3}	54-Q_{h4}	80-Q_{h5}
1	0	3	4	4	4	4	6	22	38	54	80
2	1	2	4	4	4	4	5	22	38	54	80
3	2	1	4	4	4	4	3	22	38	54	80
4	3	0	4	4	4	4	0	22	38	54	80
5	4	0	3	4	4	4	0	18	38	54	80
6	5	0	2	4	4	4	0	13	38	54	80
7	6	0	1	4	4	4	0	7	38	54	80
8	7	0	0	4	4	4	0	0	38	54	80
9	8	0	0	3	4	4	0	0	30	54	80
10	9	0	0	2	4	4	0	0	21	54	80
11	10	0	0	1	4	4	0	0	11	54	80
12	11	0	0	0	4	4	0	0	0	54	80
13	12	0	0	0	3	4	0	0	0	42	80
14	13	0	0	0	2	4	0	0	0	29	80
15	14	0	0	0	1	4	0	0	0	15	80
16	15	0	0	0	0	4	0	0	0	0	80
17	16	0	0	0	0	3	0	0	0	0	64
18	17	0	0	0	0	2	0	0	0	0	47
19	18	0	0	0	0	1	0	0	0	0	29
20	29	0	0	0	0	0	0	0	0	0	0

TABLE 7. - *Subpopulations upper means* $\overset{+}{M}_{hl}$

		Upper means				
h	y_h	$\overset{+}{M}_{h1}$	$\overset{+}{M}_{h2}$	$\overset{+}{M}_{h3}$	$\overset{+}{M}_{h4}$	$\overset{+}{M}_{h5}$
1	0	2.0	5.5	9.5	13.5	20.0
2	1	2.5	5.5	9.5	13.5	20.0
3	2	3.0	5.5	9.5	13.5	20.0
4	3	3.0	5.5	9.5	13.5	20.0
5	4	3.0	6.0	9.5	13.5	20.0
6	5	3.0	6.5	9.5	13.5	20.0
7	6	3.0	7.0	9.5	13.5	20.0
8	7	3.0	7.0	9.5	13.5	20.0
9	8	3.0	7.0	10.0	13.5	20.0
10	9	3.0	7.0	10.5	13.5	20.0
11	10	3.0	7.0	11.0	13.5	20.0
12	11	3.0	7.0	11.0	13.5	20.0
13	12	3.0	7.0	11.0	14.0	20.0
14	13	3.0	7.0	11.0	14.5	20.0
15	14	3.0	7.0	11.0	15.0	20.0
16	15	3.0	7.0	11.0	15.0	20.0
17	16	3.0	7.0	11.0	15.0	21.33
18	17	3.0	7.0	11.0	15.0	23.5
19	18	3.0	7.0	11.0	15.0	29.0
20	29	3.0	7.0	11.0	15.0	29.0

TABLE 8. - *Calculation of* $p(l|h) = \frac{P_{hl}}{P_{h.}}$

	l					Tot
	1	2	3	4	5	
h	$p(1\|h)$	$p(2\|h)$	$p(3\|h)$	$p(4\|h)$	$p(5\|h)$	1
1	1	0	0	0	0	1
2	1	0	0	0	0	1
3	1	0	0	0	0	1
4	1	0	0	0	0	1
5	0.8	0.2	0	0	0	1
6	0.6666	0.3333	0	0	0	1
7	0.5714	0.4286	0	0	0	1
8	0.5	0.5	0	0	0	1
9	0.4444	0.4444	0.1111	0	0	1
10	0.4	0.4	0.2	0	0	1
11	0.3636	0.3636	0.2727	0	0	1
12	0.3333	0.3333	0.3333	0	0	1
13	0.3077	0.3077	0.3077	0.0769	0	1
14	0.2857	0.2857	0.2857	0.1428	0	1
15	0.2666	0.2666	0.2666	0.2	0	1
16	0.25	0.25	0.25	0.25	0	1
17	0.2353	0.2353	0.2353	0.2353	0.0588	1
18	0.2222	0.2222	0.2222	0.2222	0.1111	1
19	0.2105	0.2105	0.2105	0.2105	0.1579	1
20	0.2	0.2	0.2	0.2	0.2	1

TABLE 9. - *Calculation of* $a(g|h) = \frac{1}{20-P_{h.}}\left(4 - P_{hg}\right)$ *for* $(h = 1, \ldots, 19)$ *and of* $a(g|h) = \frac{n_{hg}}{n_{h.}}$ *for* $h = 20$

	g					Tot
	1	2	3	take4	5	
h	$a(1\|h)$	$a(2\|h)$	$a(3\|h)$	$a(4\|h)$	$a(5\|h)$	1
1	0.1579	0.2105	0.2105	0.2105	0.2105	1
2	0.1111	0.2222	0.2222	0.2222	0.2222	1
3	0.0588	0.2353	0.2352	0.2353	0.2353	1
4	0	0.25	0.25	0.25	0.25	1
5	0	0.2	0.2666	0.2666	0.2666	1
6	0	0.1428	0.2857	0.2857	0.2857	1
7	0	0.0769	0.3077	0.3077	0.3077	1
8	0	0	0.3333	0.3333	0.3333	1
9	0	0	0.2727	0.3636	0.3636	1
10	0	0	0.2	0.4	0.4	1
11	0	0	0.1111	0.4444	0.4444	1
12	0	0	0	0.5	0.5	1
13	0	0	0	0.4286	0.5714	1
14	0	0	0	0.3333	0.6666	1
15	0	0	0	0.2	0.8	1
16	0	0	0	0	1	1
17	0	0	0	0	1	1
18	0	0	0	0	1	1
19	0	0	0	0	1	1
20	0	0	0	0	1	1

TABLE 10. - *Joint contributions* $\frac{1}{20}C_{hl.}$, *marginal column contributions* $\frac{1}{20}C_{h..} = \frac{1}{20}G_h$, *and marginal row contributions* $C_{.l.}$ *to the Gini* $G(Y)$ *index*

h	D_h	l					Tot.
		1	2	3	4	5	
		$\frac{1}{20}C_{h1.}$	$\frac{1}{20}C_{h2.}$	$\frac{1}{20}C_{h3.}$	$\frac{1}{20}C_{h4.}$	$\frac{1}{20}C_{h5.}$	$\frac{1}{20}G_h$
1	0.00 ⊣ 0.05	0.005	0	0	0	0	0.005
2	0.05 ⊣ 0.10	0.0095	0	0	0	0	0.0095
3	0.10 ⊣ 0.15	0.0135	0	0	0	0	0.0135
4	0.15 ⊣ 0.20	0.017	0	0	0	0	0.017
5	0.20 ⊣ 0.25	0.017	0.003	0	0	0	0.02
6	0.25 ⊣ 0.30	0.017	0.0055	0	0	0	0.0225
7	0.30 ⊣ 0.35	0.017	0.0075	0	0	0	0.0245
8	0.35 ⊣ 0.40	0.017	0.009	0	0	0	0.026
9	0.40 ⊣ 0.45	0.017	0.009	0.001	0	0	0.027
10	0.45 ⊣ 0.50	0.017	0.009	0.0015	0	0	0.0275
11	0.50 ⊣ 0.55	0.017	0.009	0.0015	0	0	0.0275
12	0.55 ⊣ 0.60	0.017	0.009	0.001	0	0	0.027
13	0.60 ⊣ 0.65	0.017	0.009	0.001	-0.001	0	0.026
14	0.65 ⊣ 0.70	0.017	0.009	0.001	-0.0025	0	0.0245
15	0.70 ⊣ 0.75	0.017	0.009	0.001	-0.0045	0	0.0225
16	0.75 ⊣ 0.80	0.017	0.009	0.001	-0.007	0	0.02
17	0.80 ⊣ 0.85	0.017	0.009	0.001	-0.007	-0.003	0.017
18	0.85 ⊣ 0.90	0.017	0.009	0.001	-0.007	-0.0065	0.0135
19	0.90 ⊣ 0.95	0.017	0.009	0.001	-0.007	-0.0105	0.0095
20	0.95 ⊣ 1.00	0.017	0.009	0.001	-0.007	-0.02	0.00
Tot	$C_{.l.}$	0.317	0.133	0.013	-0.043	-0.040	0.38 = $G(Y)$

TABLE 11. - *Joint contributions* $\frac{1}{20} V_{hl.}$, *marginal column contributions* $\frac{1}{20} V_{h..} = \frac{1}{20} V_h$, *and marginal row contributions* $B_{.l.}$ *to the Bonferroni* $V(Y)$ *index*

h	D_h	l					Tot.
		1	2	3	4	5	
		$\frac{1}{20} V_{h1.}$	$\frac{1}{20} V_{h2.}$	$\frac{1}{20} V_{h3.}$	$\frac{1}{20} V_{h4.}$	$\frac{1}{20} V_{h5.}$	$\frac{1}{20} V_h$
1	0.00 ⊣ 0.05	0.05	0	0	0	0	0.050
2	0.05 ⊣ 0.10	0.0475	0	0	0	0	0.0475
3	0.10 ⊣ 0.15	0.045	0	0	0	0	0.045
4	0.15 ⊣ 0.20	0.0425	0	0	0	0	0.0425
5	0.20 ⊣ 0.25	0.034	0.006	0	0	0	0.04
6	0.25 ⊣ 0.30	0.02833	0.00916	0	0	0	0.0375
7	0.30 ⊣ 0.35	0.024285	0.01071	0	0	0	0.035
8	0.35 ⊣ 0.40	0.02125	0.01125	0	0	0	0.0325
9	0.40 ⊣ 0.45	0.018888	0.01	0.00111	0	0	0.03
10	0.45 ⊣ 0.50	0.017	0.009	0.0015	0	0	0.0275
11	0.50 ⊣ 0.55	0.01545	0.00818	0.001363	0	0	0.025
12	0.55 ⊣ 0.60	0.014166	0.0075	0.00083	0	0	0.0225
13	0.60 ⊣ 0.65	0.01308	0.006923	0.00077	-0.00077	0	0.02
14	0.65 ⊣ 0.70	0.012142	0.006428	0.000714	-0.00178	0	0.0175
15	0.70 ⊣ 0.75	0.0113333	0.00600	0.000656	-0.003	0	0.015
16	0.75 ⊣ 0.80	0.010625	0.005625	0.000625	-0.00437	0	0.0125
17	0.80 ⊣ 0.85	0.01	0.00529	0.00059	-0.00412	-0.00176	0.01
18	0.85 ⊣ 0.90	0.00944	0.005	0.00055	-0.00388	-0.00361	0.0075
19	0.90 ⊣ 0.95	0.008945	0.004736	0.000526	-0.00368	-0.00553	0.005
20	0.95 ⊣ 1.00	0.0085	0.0045	0.0005	-0.0035	-0.010	0.00
Tot	$V_{.l.}$	0.442445	0.11632	0.00974	-0.02511	-0.02090	$0.5225 = V(Y)$

TABLE 12. - *Joint contributions* $\frac{1}{20}B_{hl.}$, *marginal column contributions* $\frac{1}{20}B_{h..} = \frac{1}{20}I_h$, *and marginal row contributions* $B_{.l.}$ *to the Zenga* $I(Y)$ *index*

h	D_h	l					Tot
		1	2	3	4	5	
		$\frac{1}{20}B_{h1.}$	$\frac{1}{20}B_{h2.}$	$\frac{1}{20}B_{h3.}$	$\frac{1}{20}B_{h4.}$	$\frac{1}{20}B_{h5.}$	$\frac{1}{20}I_h$
1	0.00 ⊣ 0.05	0.05	0	0	0	0	0.05
2	0.05 ⊣ 0.10	0.04774	0	0	0	0	0.04774
3	0.10 ⊣ 0.15	0.04569	0	0	0	0	0.04569
4	0.15 ⊣ 0.20	0.04381	0	0	0	0	0.04381
5	0.20 ⊣ 0.25	0.03526	0.00684	0	0	0	0.04211
6	0.25 ⊣ 0.30	0.02955	0.01099	0	0	0	0.04054
7	0.30 ⊣ 0.35	0.02546	0.01365	0	0	0	0.03911
8	0.35 ⊣ 0.40	0.02238	0.01541	0	0	0	0.03779
9	0.40 ⊣ 0.45	0.01999	0.01401	0.00257	0	0	0.03658
10	0.45 ⊣ 0.50	0.01806	0.01290	0.00452	0	0	0.03548
11	0.50 ⊣ 0.55	0.01649	0.01198	0.00602	0	0	0.03448
12	0.55 ⊣ 0.60	0.01517	0.01119	0.00721	0	0	0.03358
13	0.60 ⊣ 0.65	0.01406	0.01053	0.00700	0.00120	0	0.03279
14	0.65 ⊣ 0.70	0.01311	0.00996	0.00682	0.00223	0	0.03211
15	0.70 ⊣ 0.75	0.01228	0.00947	0.00667	0.00316	0	0.03158
16	0.75 ⊣ 0.80	0.01156	0.00906	0.00656	0.00406	0	0.03125
17	0.80 ⊣ 0.85	0.01093	0.00872	0.00652	0.00433	0.00074	0.03125
18	0.85 ⊣ 0.90	0.01040	0.00851	0.00662	0.00473	0.00165	0.03192
19	0.90 ⊣ 0.95	0.00998	0.00853	0.00708	0.00563	0.00327	0.03448
20	0.95 ⊣ 1.00	0.00948	0.00810	0.00672	0.00534	0.00310	0.03275
Tot	$B_{.l.}$	0.4614	0.1699	0.0743	0.0307	0.00876	$0.74506 = I(Y)$

TABLE 13. - *Decomposition by subpopulations of the Gini $G(Y)$ index $(N = 20; k = 5)$: bivariate (5×5) contributions $C_{(s)l.}$, "marginal" column contributions $C_{(s)..}$, "marginal" row contributions $C_{.l.}$, and the corresponding within and between components*

s	$D_{(s)}$		l					
			1	2	3	4	5	
1	0.0 ⊣ 0.2	$C_{(1)lW}$	0.0005	0	0	0	0	$C_{(1).W} = 0.0005$
		$C_{(1)lB}$	0.0445	0	00	0	0	$C_{(1).B} = 0.0445$
		$C_{(1)l.}$	0.045	0	0	0	0	$C_{(1)..} = 0.045$
2	0.2 ⊣ 0.4	$C_{(2)lW}$	0.000	0.0005	0	0	0	$C_{(2).W} = 0.0005$
		$C_{(2)lB}$	0.068	0.0245	0	0	0	$C_{(2).B} = 0.0925$
		$C_{(2)}l.$	0.068	0.025	0	0	0	$C_{(2)..} = 0.093$
3	0.4 ⊣ 0.6	$C_{(3)lW}$	0.000	0.000	0.0005	0	0	$C_{(3).W} = 0.0005$
		$C_{(3)lB}$	0.068	0.036	0.0045	0	0	$C_{(3).B} = 0.1085$
		$C_{(3)l.}$	0.068	0.036	0.005	0	0	$C_{(3)..} = 0.109$
4	0.6 ⊣ 0.8	$C_{(4)lW}$	0.000	0	0	0.0005	0	$C_{(4).W} = 0.0005$
		$C_{(4)lB}$	0.068	0.036	0.004	-0.0155	0	$C_{(4).B} = 0.0925$
		$C_{(4)l.}$	0.068	0.036	0.004	-0.015	0	$C_{(4)..} = 0.093$
5	0.8 ⊣ 1.0	$C_{(5)lW}$	0.000	0	0.000	0	0.002	$C_{(5).W} = 0.002$
		$C_{(5)lB}$	0.068	0.036	0.004	-0.028	-0.042	$C_{(5).B} = 0.038$
		$C_{(5)l.}$	0.068	0.036	0.004	-0.028	-0.04	$C_{(5)..} = 0.04$
	Tot	$C_{.lW}$	0.0005	0.0005	0.0005	0.0005	0.002	$C_{..W} = 0.004$
		$C_{.lB}$	0.3165	0.1325	0.0125	-0.0435	-0.042	$C_{..B} = 0.376$
		$C_{.l.}$	0.3177	0.133	0.013	-0.043	-0.040	$G(Y) = 0.38$

TABLE 14. - *Decomposition by subpopulations of the Bonferroni $V(Y)$ index ($N = 20; k = 5$): bivariate (5×5) contributions $V_{(s)l.}$, "marginal" column contributions $V_{(s)..}$, "marginal " row contributions $V_{.l.}$, and the corresponding within and between components*

s	$D_{(s)}$		l = 1	2	3	4	5	
1	0.0 ⊣ 0.2	$V_{(1)lW}$	0.003	0	0	0	0	$0.003 = V_{(1).W}$
		$V_{(1)lB}$	0.182	0	0	0	0	$0.182 = V_{(1).B}$
		$V_{(1)l.}$	0.185	0	0	0	0	$0.185 = V_{(1)..}$
2	0.2 ⊣ 0.4	$V_{(2)lW}$	0	0.00085	0	0	0	$0.00085 = V_{(2).W}$
		$V_{(2)lB}$	0.10787	0.03628	0	0	0	$0.14415 = V_{(2).B}$
		$V_{(2)l.}$	0.10787	0.03713	0	0	0	$0.145 = V_{(2)..}$
3	0.4 ⊣ 0.6	$V_{(3)lW}$	0	0	0.0005	0	0	$0.0005 = V_{(3).W}$
		$V_{(3)lB}$	0.06551	0.03468	0.00431	0	0	$0.1045 = V_{(3).B}$
		$V_{(3)l.}$	0.06551	0.03468	0.00481	0	0	$0.105 = V_{(3)..}$
4	0.6 ⊣ 0.8	$V_{(4)lW}$	0	0	0	0.00036	0	$0.00036 = V_{(4).W}$
		$V_{(4)lB}$	0.04718	0.02498	0.00276	-0.01028	0	$0.06464 = V_{(4).B}$
		$V_{(4)l.}$	0.04718	0.02498	0.00276	-0.00992	0	$0.065 = V_{(4)..}$
5	0.8 ⊣ 1.0	$V_{(5)lW}$	0	0	0	0	0.00110	$0.00110 = V_{(5).W}$
		$V_{(5)lB}$	0.03689	0.01953	0.00217	-0.01519	-0.02201	$0.0214 = V_{(5).B}$
		$V_{(5)l.}$	0.03689	0.01953	0.00217	-0.01519	-0.02096	$0.0225 = V_{(5)..}$
	Tot	$V_{.lW}$	0.003	0.00085	0.0005	0.00036	0.00110	$0.00581 = V_{..W}$
		$V_{.lB}$	0.43944	0.11546	0.00924	-0.02547	-0.02201	$0.51666 = V_{..B}$
		$V_{.l.}$	0.44244	0.11632	0.00974	-0.02512	-0.02091	$0.5225 = V(Y)$

TABLE 15. - *Decomposition by subpopulations of the Zenga $I(Y)$ index ($N = 20; k = 5$): bivariate (5×5) contributions $B_{(s)l.}$, ''marginal'' column contributions $B_{(s)..}$, ''marginal'' row contributions $B_{.l.}$, and the corresponding within and between components*

s	$D_{(s)}$		l: 1	2	3	4	5	
1	$0.0 \dashv 0.2$	$B_{(1)lW}$ $B_{(1)lB}$	0.00301 0.18423	0 0	0 0	0 0	0 0	$B_{(1).W} = 0.00301$ $B_{(1).B} = 0.18423$
		$B_{(1)l.}$	0.18724	0	0	0	0	$B_{(1)..} = 0.18724$
2	$0.2 \dashv 0.4$	$B_{(2)lW}$ $B_{(2)lB}$	0 0.11265	0.00092 0.04597	0 0	0 0	0 0	$B_{(2).W} = 0.00091$ $B_{(2).B} = 0.15863$
		$B_{(2)l.}$	0.11265	0.04689	0	0	0	$B_{(2)..} = 0.15954$
3	$0.4 \dashv 0.6$	$B_{(3)lW}$ $B_{(3)lB}$	0 0.06971	0 0.05009	0.00065 0.01967	0 0	0 0	$B_{(3).W} = 0.00065$ $B_{(3).B} = 0.13948$
		$B_{(3)l.}$	0.06971	0.05009	0.02032	0	0	$B_{(3)..} = 0.14013$
4	$0.6 \dashv 0.8$	$B_{(4)lW}$ $B_{(4)lB}$	0 0.05108	0 0.03903	0 0.02704	0.00066 0.01000	0 0	$B_{(4).W} = 0.00066$ $B_{(4).B} = 0.12714$
		$B_{(4)l.}$	0.05108	0.03903	0.02704	0.01065	0	$B_{(4)..} = 0.12779$
5	$0.8 \dashv 1.0$	$B_{(5)lW}$ $B_{(5)lB}$	0 0.0408	0 0.03386	0 0.02694	0 0.02003	0.00876 0	$B_{(5).W} = 0.00876$ $B_{(5).B} = 0.12164$
		$B_{(5)l.}$	0.0408	0.03386	0.02694	0.02003	0.00876	$B_{(5)..} = 0.13040$
	Tot	$B_{.lW}$ $B_{.lB}$	0.00301 0.45841	0.00091 0.16896	0.00065 0.07366	0.00066 0.03003	0.00876 0.0000?	$B_{..W} = 0.01400$ $B_{..B} = 0.73105$
		$B_{.l.}$	0.46149	0.16987	0.07431	0.03068	0.00876	$I(Y) = 0.74505$

From the above reported relation $V(Y) = \sum_{s=1}^{5} V_{(s)..}$ we obtain:

$$V(Y) = \sum_{s=1}^{5} V_{(s)..} = \sum_{s=1}^{5} \left[\sum_{h=1+4(s-1)}^{4s} \frac{1}{20} V_h \right]$$

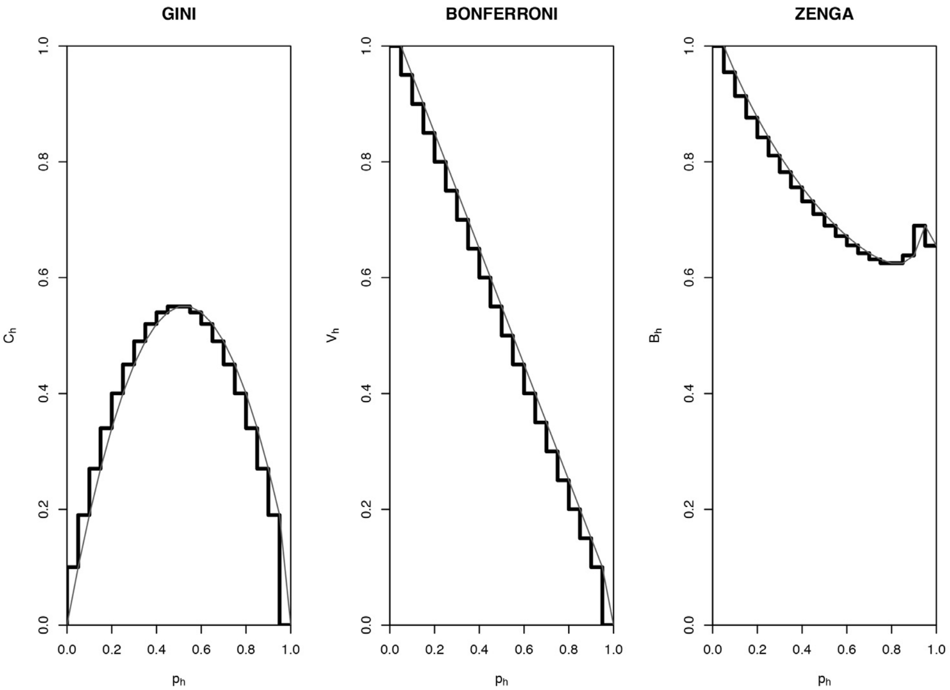
$$= \sum_{s=1}^{5} \left[\frac{4}{20} \left(\frac{1}{4} \sum_{h=1+4(s-1)}^{4s} V_h \right) \right] = \sum_{s=1}^{5} 0.2 \cdot \overline{V}_s,$$

where

$$\overline{V}_s = \frac{1}{4} \cdot \sum_{h=1+4(s-1)}^{4s} V_h, \qquad s = 1, ..., 5$$

is the mean of the four Bonferroni point measures V_h, $(1 + 4(s-1) \leq h \leq 4 \cdot s)$. In other words, we can reppresent the value of $V(Y)$ as the sum of the areas of 5 rectangles with bases 0.2 and heights $\overline{V}_s$. This kind of representation can be extended to the Gini and Zenga indexes as illustrated in Figure 1. We conclude this section pointing out that the aggregation of the units in groups delimitated by the quintiles (quartiles,...) has no effect on the values of the contributions $(V_{.l.}, C_{.l.}, B_{.l.})$ of the subpopulations to the three synthetic indexes.

FIGURE 1. - *Graphs of Gini, Bonferroni and Zenga inequality curves*



4.2 *Numerical example of a population of $N = 7$ units partitioned into $K = 3$ non overlapping subpopulations*

In this second example the variate Y assumes the following $r = 6$ distinct values $(y_1 = 1, y_2 = 3, y_3 = 4, y_4 = 6, y_5 = 7, y_6 = 10)$, and $(n_{1.} = 2; n_{2.} = n_{3.} = n_{4.} = = n_{5.} = n_{6.} = 1)$ are the corresponding frequencies. The sizes of the three subpopulations are: $n_{.1} = 2, n_{.2} = 3, n_{.3} = 2$. In Table 17 we have reported: the 6×3 bivariate frequency distribution of the example at hand, the lower and upper group frequencies, and, the corresponding lower and upper group incomes. Then, in Table 18 we have reported the lower and the upper means for each subpopulation and for the whole population. In Table 19 we have reported the calculation of the three point measures (I_h, V_h, G_h) and of the three synthetic inequality indexes

$[I(Y), V(Y), G(Y)]$. In Table 20 we have reported the relative frequencies $p(l|h)$ of the subpopulation l in the lower group $(Y \leq y_h)$ and the relative frequencies $a(l|h)$ of the subpopulation l in the upper group $(Y > y_h)$. Then, in Tables 21, 22, and 23 we have reported the decompositions by subpopulations of the Gini, Bonferroni and Zenga synthetic index, respectively. Now we provide some details on the calculation of the values reported in these latter Tables. The first row $(h = 1)$ of Table 22 reports the joint contributions $(n_{1.}/7) \cdot V_{1l.}$, $(l = 1, 2, 3)$, and the ''marginal'' column contribution $(n_{1.}/7) \cdot \sum_{l=1}^{3} V_{1l.} = (n_{1.}/7) \cdot V_{1..} = = (n_{1.}/7) \cdot V_1 = (2/7) \cdot 0.78125 =$ =0.223214. The contribution of subpopulation l to the Bonferroni point index is given by: $V_{1l.} = \frac{M - \bar{M}_{hl.}}{M} p(l|h)$. For the present example, where $M = (32/7)$, we report the following Table 16 for the related calculus:

TABLE 16. - *Calculation related the first row* $(h = 1)$ *of Table 22*

l	$\bar{M}_{1l}$	$\frac{M-\bar{M}_{1l}}{M}$	$p(l\|h)$	$V_{1l.}$	$\frac{2}{7} V_{1l.}$
1	1	0.78125	0.5	0.390625	0.111607
2	1	0.78125	0.5	0.390625	0.111607
3	6	-0.3125	0	0	0
Total			1	0.78125=	0.223214=
				$V_{1..} = V_1$	$\frac{2}{7} V_{1..} = \frac{2}{7} V_1$

In addition, for the Bonferroni index, we give in Table 24 some details on: the calculation for $(l = 1, 2, 3)$ of $(n_{4.}/7) \cdot V_{4lW}$, $(n_{4.}/7) \cdot V_{4lB}$ and $(n_{4.}/7) \cdot V_{4l.}$, and, the calculation of $(n_{4.}/7) \cdot V_{4.W}$, $(n_{4.}/7) \cdot V_{4.B}$ and $(n_{4.}/7) \cdot V_{4..} = (n_{4.}/7) \cdot V_4$. In Table 25 we extend these ''details'' to the Zenga index, too.

TABLE 17. - *Joint frequencies n_{hl}, total frequencies $n_{.l}$ and $n_{h.}$ of a population of $N = 7$ units partitioned into $k = 3a$ subpopulations. Lower group frequencies P_{hl} and Lower group incomes Q_{hl}, upper group frequencies $(n_{.l} - P_{hl})$ and upper group incomes $(T_l - Q_{hl})$. Values of the functions $o(l)$ and $u(l)$*

n_{hl}		l			$n_{h.}$								
h	y_h	1	2	3		P_{h1}	P_{h2}	P_{h3}	$P_{h.}$	Q_{h1}	Q_{h2}	Q_{h3}	$Q_{h.}$
1	1	1	1	0	2	1	1	0	2	1	1	0	2
2	3	1	0	0	1	2	1	0	3	4	1	0	5
3	4	0	1	0	1	2	2	0	4	4	5	0	9
4	6	0	0	1	1	2	2	1	5	4	5	6	15
5	7	0	1	0	1	2	3	1	6	4	12	6	22
6	10	0	0	1	1	2	3	2	7	4	12	16	32
$n_{.l}$		2	3	2	7								
$o(l)$		1	1	4		$2-P_{h1}$	$3-P_{h2}$	$2-P_{h3}$	$7-P_{h.}$	$4-Q_{h1}$	$12-Q_{h2}$	$16-Q_{h3}$	$32\ Q_{h.}$
$u(l)$		2	5	6									
						1	2	2	5	3	11	16	30
						0	2	2	4	0	11	16	27
						0	1	2	3	0	7	16	23
						0	1	1	2	0	7	10	17
						0	0	1	1	0	0	10	10
						0	0	0	0	0	0	0	0

TABLE 18. - *Lower means $(\bar{M}_{hl}, \bar{M}_{h.})$ and Upper means $(\overset{+}{M}_{hl}, \overset{+}{M}_{h.})$*

		Lower means					Upper means			
h	y_h	$\bar{M}_{h1}$	$\bar{M}_{h2}$	$\bar{M}_{h3}$	$\bar{M}_{h.}$		$\overset{+}{M}_{h1}$	$\overset{+}{M}_{h2}$	$\overset{+}{M}_{h.3}$	$\overset{+}{M}_{h.}$
1	1	1	1	6	1		3	5.5	8	6
2	3	2	1	6	1.666		3	5.5	8	6.75
3	4	2	2.5	6	2.25		3	7	8	7.666
4	6	2	2.5	6	3		3	7	10	8.5
5	7	2	4	6	3.666		3	7	10	10
6	10	2	4	8	32/7		3	7	10	10

TABLE 19. - *Calculation of* $I_h = \frac{\overset{+}{M}_{h.} - \bar{M}_{h.}}{\overset{+}{M}_{h.}}$, $V_h = \frac{M - \bar{M}_{h.}}{M}$, *and* $G_h = (V_h \cdot 2p_{h.} - \frac{n_{h.}-1}{7} \cdot \frac{M - y_h}{M})$ *point inequality measures*

h	I_h	V_h	$2p_{h.}$	$\frac{n_{h.}-1}{7}\frac{M-y_h}{M}$	G_h	$I_h \cdot \frac{n_{h.}}{7}$	$V_h \cdot \frac{n_{h.}}{7}$	$G_h \cdot \frac{n_{h.}}{7}$
1	0.833333	0.78125	0.571428	0.111607	0.334821	0.238095	0.223214	0.095663
2	0.753086	0.635416	0.857143	0	0.544643	0.107584	0.090774	0.077806
3	0.706522	0.507812	1.142857	0	0.580357	0.100931	0.072544	0.082908
4	0.647059	0.34375	1.428571	0	0.491071	0.092437	0.049107	0.070153
5	0.633333	0.197917	1.714286	0	0.339286	0.090476	0.028274	0.048469
6	0.542857	0	2	0	0	0.077551	0	0
						0.707075 $= I(Y)$	0.463914 $= V(Y)$	0.375 $= G(Y)$

TABLE 20. - *Calculation of* $p(l|h) = \frac{P_{hl}}{P_{h.}}$. *Calculation of* $a(l|h) = \frac{1}{7-P_{h.}}(n_{.l} - P_{hl})$ *for* $h = 1, \ldots, 5$, *and of* $a(l|h) = n_{6l}/n_{6.}$ *for* $h = 6$

	l			Tot		l			Tot
	1	2	3			1	2	3	
h	$p(1\|h)$	$p(2\|h)$	$p(3\|h)$	1		$a(1\|h)$	$a(2\|h)$	$a(3\|h)$	1
1	0.5	0.5	0	1		0.2	0.4	0.4	1
2	0.6666	0.3333	0	1		0	0.5	0.5	1
3	0.5	0.5	0	1		0	0.3333	0.6666	1
4	0.4	0.4	0.2	1		0	0.5	0.5	1
5	0.3333	0.5	0.1666	1		0	0	1	1
6	2/7	3/7	2/7	1		0	0	1	1

TABLE 21. - *Decomposition by subpopulations of the Gini $G(Y)$ index ($N = 7, k = 3$): joint (6×3) contributions $\frac{n_{h.}}{7} C_{hl.}$, ''marginal column contributions $\frac{n_{h.}}{7} C_{h..} = \frac{n_{h.}}{7} G_h$, ''marginal'' row contributions $C_{.l.}$, and theirs within and between parts*

h	D_h		l = 1	2	3	
1	$0 \dashv \frac{2}{7}$	$C_{1lW}(2/7)$	0.003827	0.017219	0	$C_{1.W}(2/7) = 0.021046$
		$C_{1lB}(2/7)$	0.044004	0.030612	0	$C_{1.W}(2/7) = 0.074616$
		$C_{1l.}(2/7)$	0.047831	0.047831	0	$C_{1..}(2/7) = 0.095663$
2	$\frac{2}{7} \dashv \frac{3}{7}$	$C_{2lW}(1/7)$	0	0.011480	0	$C_{2.W}(1/7) = 0.011480$
		$C_{2lB}(1/7)$	0.045918	0.020408	0	$C_{2.B}(1/7) = 0.066326$
		$C_{2l.}(1/7)$	0.045918	0.031888	0	$C_{2..}(1/7) = 0.077806$
3	$\frac{3}{7} \dashv \frac{4}{7}$	$C_{3lW}(1/7)$	0	0.011480	0	$C_{3.W}(1/7) = 0.011480$
		$C_{3lB}(1/7)$	0.045918	0.025510	0	$C_{3.B}(1/7) = 0.071428$
		$C_{3l.}(1/7)$	0.045918	0.036990	0	$C_{3..}(1/7) = 0.082908$
4	$\frac{4}{7} \dashv \frac{5}{7}$	$C_{4lW}(1/7)$	0	0.011480	0.005102	$C_{4.W}(1/7) = 0.016582$
		$C_{4lB}(1/7)$	0.045918	0.025510	-0.017857	$C_{4.B}(1/7) = 0.053571$
		$C_{41l.}(1/7)$	0.045918	0.036990	-0.012755	$C_{4..}(1/7) = 0.070153$
5	$\frac{5}{7} \dashv \frac{6}{7}$	$C_{5lW}(1/7)$	0	0	0.005102	$C_{5.W}(1/7) = 0.005102$
		$C_{5lB}(1/7)$	0.045918	0.015306	-0.017857	$C_{5.B}(1/7) = 0.043367$
		$C_{5l.}(1/7)$	0.045918	0.015306	-0.012755	$C_{5..}(1/7) = 0.048469$
6	$\frac{6}{7} \dashv 1$	$C_{6lW}(1/7)$	0	0	0	$C_{6.W}(1/7) = 0$
		$C_{6lB}(1/7)$	0.045918	0.015306	-0.061223	$C_{6.B}(1/7) = 0$
		$C_{6l.}(1/7)$	0.045918	0.015306	-0.061223	$C_{6..}(1/7) = 0$
	Tot	$C_{.lW}$	0.003827	0.05166	0.010204	$C_{..W} = 0.065690$
		$C_{.lB}$	0.273596	0.132651	-0.096937	$C_{..B} = 0.309310$
		$C_{.l.}$	0.277423	0.184311	-0.086733	$G(Y) = 0.375$

TABLE 22. - *Decomposition by subpopulations of the Bonferroni $V(Y)$ index $(N = 7, k = 3)$: joint (6×3) contributions $\frac{n_{h.}}{7} V_{hl.}$, ``marginal column contributions $\frac{n_{h.}}{7} V_{h..} = \frac{n_{h.}}{7} V_h$, ``marginal'' row contributions $V_{.l.}$, and theirs within and between parts*

h	D_h		l			
			1	2	3	
1	$0 \dashv \frac{2}{7}$	$V_{1lW}(2/7)$	0.008929	0.040178	0	$V_{1.W} \cdot (n_{h.}/7) = 0.049107$
		$V_{1lB}(2/7)$	0.102678	0.071429	0	$V_{1.W} \cdot (n_{h.}/7) = 0.174107$
		$V_{1l.}(2/7)$	0.111607	0.111607	0	$V_{1..} \cdot (n_{h.}/7) = 0.223214$
2	$\frac{2}{7} \dashv \frac{3}{7}$	$V_{2lW}(1/7)$	0	0.013393	0	$V_{2.W} \cdot (n_{h.}/7) = 0.013393$
		$V_{2lB}(1/7)$	0.053571	0.023809	0	$V_{2.B} \cdot (n_{h.}/7) = 0.077380$
		$V_{2l.}(1/7)$	0.053571	0.037202	0	$V_{2..} \cdot (n_{h.}/7) = 0.090774$
3	$\frac{3}{7} \dashv \frac{4}{7}$	$V_{3lW}(1/7)$	0	0.010045	0	$V_{3.W} \cdot (n_{h.}/7) = 0.010045$
		$V_{3lB}(1/7)$	0.040179	0.022321	0	$V_{3.B} \cdot (n_{h.}/7) = 0.062500$
		$V_{3l.}(1/7)$	0.040179	0.032366	0	$V_{3..} \cdot (n_{h.}/7) = 0.072545$
4	$\frac{4}{7} \dashv \frac{5}{7}$	$V_{4lW}(1/7)$	0	0.008036	0.003571	$V_{4.W} \cdot (n_{h.}/7) = 0.011607$
		$V_{4lB}(1/7)$	0.032143	0.017857	-0.012500	$V_{4.B} \cdot (n_{h.}/7) = 0.037500$
		$V_{41l.}(1/7)$	0.032143	0.025893	-0.008929	$V_{4..} \cdot (n_{h.}/7) = 0.049107$
5	$\frac{5}{7} \dashv \frac{6}{7}$	$V_{5lW}(1/7)$	0	0	0.002976	$V_{5.W} \cdot (n_{h.}/7) = 0.002976$
		$V_{5lB}(1/7)$	0.026786	0.008929	-0.010416	$V_{5.B} \cdot (n_{h.}/7) = 0.025299$
		$V_{5l.}(1/7)$	0.026786	0.008929	-0.007440	$V_{5..} \cdot (n_{h.}/7) = 0.028274$
6	$\frac{6}{7} \dashv 1$	$V_{6lW}(1/7)$	0	0	0	$V_{6.W} \cdot (n_{h.}/7) = 0$
		$V_{6lB}(1/7)$	0.022959	0.007653	-0.030611	$V_{6.B} \cdot (n_{h.}/7) = 0$
		$V_{6l.}(1/7)$	0.022959	0.007653	-0.030611	$V_{6..} \cdot (n_{h.}/7) = 0$
	Tot	$V_{.lW}$	0.008929	0.071652	0.006547	$V_{..W} = 0.087124$
		$V_{.lB}$	0.278316	0.151998	-0.053527	$V_{..B} = 0.376790$
		$V_{.l.}$	0.287245	0.223650	-0.046980	$V(Y) = 0.463914$

TABLE 23. - *Decomposition by subpopulations of the Zenga $I(Y)$ index ($N = 7, k = 3$): joint (6×3) contributions $\frac{n_{h.}}{7} B_{hl.}$, "marginal column contributions $\frac{n_{h.}}{7} B_{h..} = \frac{n_{h.}}{7} I_h$, "marginal" row contributions $B_{.l.}$, and theirs within and between parts*

h	D_h		l = 1	2	3	
1	$0 \dashv \frac{2}{7}$	$B_{1lW}(2/7)$	0.009524	0.042857	0	$B_{1.W}(2/7) = 0.052381$
		$B_{1lB}(2/7)$	0.109524	0.076190	0	$B_{1.B}(2/7) = 0.185714$
		$B_{1l.}(2/7)$	0.119047	0.119047	0	$B_{1..}(2/7) = 0.238095$
2	$\frac{2}{7} \dashv \frac{3}{7}$	$B_{2lW}(1/7)$	0	0.015873	0	$B_{2.W}(1/7) = 0.015873$
		$B_{2lB}(1/7)$	0.067019	0.024692	0	$B_{2.B}(1/7) = 0.091711$
		$B_{2l.}(1./7)$	0.067019	0.040565	0	$B_{2..}(1/7) = 0.107584$
3	$\frac{3}{7} \dashv \frac{4}{7}$	$B_{3lW}(1/7)$	0	0.013975	0	$B_{3.W}(n_{h.}/7) = 0.013975$
		$B_{3lB}(1/7)$	0.052795	0.034162	0	$B_{3.B}(n_{h.}/7) = 0.086957$
		$B_{3l.}(1/7)$	0.052795	0.048137	0	$B_{3..}(n_{h.}/7) = 0.100932$
4	$\frac{4}{7} \dashv \frac{5}{7}$	$B_{4lW}(1/7)$	0	0.015126	0.006722	$B_{4.W}(n_{h.}/7) = 0.021849$
		$B_{4lB}(1/7)$	0.043697	0.025210	0.001681	$B_{4.B}(n_{h.}/7) = 0.070588$
		$B_{4l.}(1/7)$	0.043697	0.040336	0.008403	$B_{4..}(n_{h.}/7) = 0.092437$
5	$\frac{5}{7} \dashv \frac{6}{7}$	$B_{5lW}(1/7)$	0	0	0.009524	$B_{5.W}(n_{h.}/7) = 0.009524$
		$B_{5lB}(1/7)$	0.038095	0.042857	0	$B_{5.B}(n_{h.}/7) = 0.080952$
		$B_{5l.}(1/7)$	0.038095	0.042857	0.009524	$B_{5..}(n_{h.}/7) = 0.090476$
6	$\frac{6}{7} \dashv 1$	$B_{6lW}(1/7)$	0	0	0.008163	$B_{6.W}(n_{h.}/7) = 0.008163$
		$B_{6lB}(1/7)$	0.032653	0.036735	0	$B_{6.B}(n_{h.}/7) = 0.069388$
		$B_{6l.}(1/7)$	0.032653	0.036735	0.008163	$B_{6..}(n_{h.}/7) = 0.077551$
	Tot	$B_{.lW}$	0.009523	0.08783	0.024401	$B_{..W} = 0.121764$
		$B_{.lB}$	0.343784	0.239845	0.001689	$B_{..B} = 0.585311$
		$B_{.l.}$	0.353307	0.327676	0.026090	$I(Y) = 0.707075$

TABLE 24. - *Calculation of:* $\frac{n_{4.}}{7}V_{4lW}=\frac{1}{7}\,[(M_l-\bar{M}_{4l})/M]p(l|4)\,\frac{n_{.l}}{7}$; $\frac{n_{4.}}{7}V_{4lB}=\frac{1}{7}\sum_{g\neq l}[(M_g-\bar{M}_{4l})/M]p(l|4)\,\frac{n_{.g}}{7}$; $\frac{n_{4.}}{7}V_{4l.}=\frac{1}{7}\,[(M-\bar{M}_{4l})/M]p(l|4)$

$h=4, M=\frac{32}{7}$	$l=1$ $\bar{M}_{4\,1}=2$	$l=2$ $\bar{M}_{4\,2}=2.5$	$l=3$ $\bar{M}_{4\,3}=6$	$\frac{n_{.g}}{7}$
$g=1; M_1=2$	$\frac{1}{7}\frac{2-2}{M}0.4\frac{3}{7}$ $=0.0$	$\frac{1}{7}\frac{2-2.5}{M}0.4\frac{3}{7}$ $=\frac{1}{7}(-0.0125)$ $=-0.001786$	$\frac{1}{7}\frac{2-6}{M}0.2\frac{2}{7}$ $=\frac{1}{7}(-0.05)$ $=-0.007143$	$\frac{2}{7}$
$g=2; M_2=4$	$\frac{1}{7}\frac{4-2}{M}0.4\frac{3}{7}$ $=\frac{1}{7}0.075$ $=0.010714$	$\frac{1}{7}\frac{4-2.5}{M}0.4\frac{3}{7}$ $=\frac{1}{7}0.05625$ $=0.008036$	$\frac{1}{7}\frac{4-6}{M}0.2\frac{3}{7}$ $=\frac{1}{7}(-0.0375)$ $=-0.005371$	$\frac{3}{7}1-1$
$g=3; M_3=8$	$\frac{1}{7}\frac{8-2}{M}0.4\frac{2}{7}$ $=\frac{1}{7}0.15$ $=0.021428$	$\frac{1}{7}\frac{8-2.5}{M}0.4\frac{2}{7}$ $=\frac{1}{7}0.1375$ $=0.019643$	$\frac{1}{7}\frac{8-6}{M}0.2\frac{2}{7}$ $=\frac{1}{7}0.025$ $=0.003571$	$\frac{2}{7}$
$p(l \mid 4)$	0.4	0.4	0.2	1

$V_{4lW}(1/7)$	0	0.008036	0.003571	$0.0116=V_{4.W}\frac{1}{7}$
$V_{4lB}(1/7)$	0.032143	0.017857	-0.01250	$0.0375=V_{4.B}\frac{1}{7}$
$V_{4l.}(1/7)$	0.032143	0.02589	-0.008929	$0.0491=V_{4..}\frac{1}{7}$

TABLE 25. - *Calculation of:* $\frac{n_{4.}}{7}B_{4lW}=\frac{1}{7}\frac{\overset{+}{M}_{4l}-\bar{M}_{4l}}{\overset{+}{M}_{4.}}p(l|4)a(l|4)$;

$$\frac{n_{4.}}{7}B_{4lB}=\frac{1}{7}\sum_{g\neq l}\frac{\overset{+}{M}_{4g}-\bar{M}_{4l}}{\overset{+}{M}_{4.}}p(l|4)a(g|4);$$

$$\frac{1}{7}B_{4lW}+\frac{1}{7}B_{4lB}=\frac{1}{7}B_{4l.}=\frac{1}{7}\frac{\overset{+}{M}_{4.}-\bar{M}_{4l}}{\overset{+}{M}_{4.}}p(l|4)=\frac{1}{7}\sum_{g=1}^{3}\frac{\overset{+}{M}_{4g}-\bar{M}_{4l}}{\overset{+}{M}_{4.}}p(l|4)a(g|4)$$

$h=4; \overset{+}{M}_{4.}=8.5$ $\frac{n_{4.}}{N}=\frac{1}{7}$	$l=1$ $\bar{M}_{41}=2$	$l=2$ $\bar{M}_{42}=2.5$	$l=3$ $\bar{M}_{43}=6$	$a(g\|4)=\frac{n_{.g}-P_{4g}}{7-P_{4.}}$
$g=1; \overset{+}{M}_{41}=3$	0	0	0	0
$g=2; \overset{+}{M}_{42}=7$	$\frac{1}{7}\frac{7-2}{8.5}0.4\cdot 0.5$ $=\frac{1}{7}.0.117647$ $=0.0168067$	$\frac{1}{7}\frac{7-2.5}{8.5}0.4\cdot 0.5$ $=\frac{1}{7}0.10588$ $=0.015126$	$\frac{1}{7}\frac{7-6}{8.5}0.2\cdot 0.5$ $=\frac{1}{7}0.0117647$ $=0.001681$	0.5
$g=3; \overset{+}{M}_{43}=10$	$\frac{1}{7}\frac{10-2}{8.5}0.4\cdot 0.5$ $=\frac{1}{7}\cdot 0.18823$ $=0.026891$	$\frac{1}{7}\frac{10-2,5}{8.5}0.4\cdot 0.5$ $=\frac{1}{7}0.176471$ $=0.025210$	$\frac{1}{7}\frac{10-6}{8.5}0.2\cdot 0.5$ $=\frac{1}{7}0.047058$ $=0.006723$	0.5
$p(l \mid 4)$	0.4	0.4	0.2	1

$B_{4lW}(1/7)$	0	0.01513	0.006723	$0.021849=B_{4.W}\frac{1}{7}$
$B_{4lB}(1/7)$	0.043697	0.02521	0.001681	$0.070588=B_{4.B}\frac{1}{7}$
$B_{4l.}(1/7)a$	0.043697	0.040336	0.008404	$0.092437=B_{4..}\frac{1}{7}$

5. FINAL REMARKS AND CONCLUSIONS

The Gini $G(Y)$, Bonferroni $V(Y)$, and Zenga $I(Y)$ synthetic inequality indexes are the wheighted arithmetic mean of the corresponding point indexes $G_h(Y)$, $V_h(Y)$ and $I_h(Y)$: $G(Y) = \sum_{h=1}^{r} G_h(Y) \cdot (n_{h.}/N)$, $V(Y) = \sum_{h=1}^{r} V_h(Y) \cdot (n_{h.}/N)$, $I(Y) = = \sum_{h=1}^{r} I_h(Y) \cdot (n_{h.}/N)$; $(y_1 < ... < y_h < ... < y_r)$ is the set of the r distinct values assumed by Y over the k subpopulations and $(n_{1.}, ..., n_{h.}, ..., n_{r.} : \sum_{h=1}^{r} n_{h.} = N)$ are the corresponding frequencies. At each y_h the whole population is split into two non overlapping groups: a lower group $(Y \leq y_h)$ including the first $P_{h.} = \sum_{t=1}^{h} n_{t.}$ units and the corresponding upper group $(Y > y_h)$ including the remaining $(N - P_{h.})$ units. M, $\bar{M}_{h.}$, and $\overset{+}{M}_{h.}$ are the arithmetic means of the whole population, of the lower group and of the upper group, respectively. The three point indexes are given by: $I_h = (\overset{+}{M}_{h.} - \bar{M}_{h.})/\overset{+}{M}_{h.}$, $V_h = (M - \bar{M}_{h.})/M$, and $G_h = \frac{M-\bar{M}_{h.}}{M} 2p_{h.} - -\left(\frac{n_{h.}-1}{N}\right)\left(\frac{M-y_h}{M}\right)$; $p_{h.} = \frac{P_{h.}}{N}$. For the decomposition by subpopulations of the three indexes, the $r \times k$ bivariate distribution (Table 1) of the N units according to the k subpopulations and to the r distinct values of Y plays an important role. In particular: n_{hg} is the frequency of y_h in the subpopulation $g \in (1, ..., k)$, $n_{.g}$ is the size of subpopulation g , and $N = \sum_{g=1}^{k} n_{.g}$. For the distribution $\{(y_h, n_{hg}) : h = 1, ..., r\}$ of the subpopulation g : $P_{hg} = \sum_{t=1}^{h} n_{tg}$ and $(n_{.g} - P_{hg})$ are the units of subpopulation g with $(Y \leq y_h)$ and $(Y > y_h)$ respectively, and, $\bar{M}_{hg}$ and $\overset{+}{M}_{hg}$ are the corresponding lower and upper means. Note that M_g denotes the mean of subpopulation g.

The decompositions illustrated in this paper are related to the decompositions proposed by Zenga (2016a, 2016b), and Zenga and Valli (2016). The lower mean $\bar{M}_{h.}$ of the whole population is related to the corresponding lower means of the k subpopulations by the relation $\bar{M}_{h.} = \sum_{l=1}^{k} \bar{M}_{hl} \cdot p(l|h)$, where $p(l|h) = (P_{hl}/P_{h.})$ is the relative frequency of subpopulation l in the lower group $(Y \leq y_h)$. Putting the above reported relation in the numerator of I_h and of V_h, the following additive decompositions of Zenga and Bonferroni point indexes are obtained: $I_h = \sum_{l=1}^{k} \frac{\overset{+}{M}_{h.} - \bar{M}_{hl.}}{\overset{+}{M}_{h.}} p(l|h) = \sum_{l=1}^{k} B_{hl.}$; $V_h = \sum_{l=1}^{k} \frac{M - \bar{M}_{hl.}}{M} p(l|h) = \sum_{l=1}^{k} V_{hl}$. The decomposition by subpopulations of the Gini point indes is given by:

$$G_h = \sum_{l=1}^{k} \left[\frac{M - \bar{M}_{hl.}}{M} p(l|h) 2p_h. - \frac{M - y_h}{M} \frac{n_{h.} - 1}{N} f(l|h) \right]$$
$$= \sum_{l=1}^{k} \left[V_{hl.} 2p_h. - \frac{M - y_h}{M} \frac{n_h. - 1}{N} f(l \mid h) \right]$$
$$= \sum_{h=1}^{k} C_{hl.}$$

where, $f(l|h) = \frac{n_{hl}}{n_{h.}}$. Putting these (three) decompositions in the expressions of the corresponding synthetic indexes, the following $r \times k$ ‘‘bivariate decompositions’’ are obtained:

$$I(Y) = \sum_{h=1}^{r} I_h \cdot (n_{h.}/N) = \sum_{h=1}^{r}\sum_{l=1}^{k} B_{hl.}(n_{h.}/N) = \sum_{l=1}^{k}\sum_{h=1}^{r} B_{hl.}(n_{h.}/N) = \sum_{l=1}^{k} B_{.l.};$$

$$V(Y) = \sum_{h=1}^{r} V_h \cdot (n_{h.}/N) = \sum_{h=1}^{r}\sum_{l=1}^{k} V_{hl.}(n_{h.}/N) = \sum_{l=1}^{k}\sum_{h=1}^{r} V_{hl.}(n_{h.}/N) = \sum_{l=1}^{k} V_{.l.};$$

$$G(Y) = \sum_{h=1}^{r} G_h \cdot (n_{h.}/N) = \sum_{h=1}^{r}\sum_{l=1}^{k} C_{hl.}(n_{h.}/N) = \sum_{l=1}^{k}\sum_{h=1}^{r} C_{hl.}(n_{h.}/N) = \sum_{l=1}^{k} C_{.l.}.$$

We remark that, the decompositions illustrated in this paper can be utilized in the case of non-overlapping subpopulations as well as in the overlapping case. Thus, the decompositions of the three indexes are illustrated by a numerical example of a population of $N = 20$ units partitioned into $k = 5$ non-overlapping subpopulations and by a numerical example of a population of $N = 7$ units partitioned into $k = 3$ overlapping subpopulations. In the first example the variate Y assumes the following $r = 20$ distinct values $(y_1 = 0; y_2 = 1; ...; y_{19} = 18; y_{20} = 29)$, and, the supports of the 5 non-overlapping subpopulations are: $[y_1 = 0; ...; y_4 = 3], [y_5 = 4; ...; y_8 = 7], [y_9 = 8; ...; y_{12} = 11], [y_{13} = 12; ...; y_{16} = 15], [y_{17} = 16; ...; y_{19} = 18; y_{20} = 29]$. The means of the whole population and of the 5 subpopulations are: $M = 10$; $M_1 = 1.5$; $M_2 = 5.5$; $M_3 = 9.5$; $M_4 = 13.5$; $M_5 = 20$. Table 3 reports, the joint frequencies n_{hl}, and the corresponding marginal freqencies $n_{h.} = 1, (h = 1, ..., 20)$ and $n_{.l} = 4, (l = 1, ..., 5)$. Table 4 reports the calculation of the three point indexes and of the corresponding synthetic measures. In the present example for each h, $(n_{h.} - 1) = 0$, consequently $G_h = V_h \cdot 2p_{h.}$. Figure 1 displays the ''graphs'' of I_h, V_h and G_h. Note that in these ''graphs'' the values of $I(Y), V(Y)$ and $G(Y)$ are equal to the sum of the areas of r rectangles , with bases $n_{h.}/N$ and heights I_h, V_h and G_h, respectively. Finally, Tables 10, 11 and 12 report for each synthetic index $[G(Y), V(Y), I(Y)]$ the 20×5 bivariate contributions $[\frac{1}{20} C_{hl.}, \frac{1}{20} V_{hl.}, \frac{1}{20} B_{hl.}]$, and the corresponding row and column marginal contributions $\left[\left(C_{.l.}, \frac{1}{20} G_h\right), \left(V_{.l.}, \frac{1}{20} V_h\right), \left(B_{.l.}, \frac{1}{20} I_h\right)\right]$. It is worth remarking that in present numerical example (five non overlapping subpopulations) there are (Tables: 10,11,12) 40 bivariate contributions that are equal to zero. This happens because 40 relative frequencies $p(l \mid h) = \frac{P_{hl}}{P_{h.}} = \frac{0}{P_{h.}} = 0$: see Tables 3 and 8. In addition we remark that for the Bonferroni and the Gini indexes (Tables 10, 11) there are eight negative bivariate contributions for the subpopulation $l = 4$, and 4 negative bivariate contributions for the subpopulation $l = 5$. Viceversa for the Zenga index the corresponding contributions are positive. In Table 26 we provide the calculation of $[\frac{1}{20} V_{h4.}, V_{.4.}]$ and of $[\frac{1}{20} B_{h4.}, B_{.4.}]$.

In applied analysis it may be useful to aggregate the N units in groups delimitated by quantiles (e.g. quartiles or quintiles). In the example with $N = 20$ units, using the quintiles the five groups are $\left[y_{1+4(s-1)}, ..., y_{4s}; s = 1, ..., 5\right]$. Note that these groups are the $k = 5$ subpopulations whose ''contributions'' are reported in the (20×5) bivariate Tables (10, 11, and 12). In particular, the h^{th} row of Table 11 reports the five contributions $(n_{h.}/20)V_{hl.}$ and the marginal contribution $(n_{h.}/20)V_{h..} = (n_{h.}/20)V_h$. We point out that these six contributions are ''related'' to the interval $D_h = (p_{h-1.} \dashv p_{h.})$ of the Bonferroni graph. In the case of the quintiles we have aggregated the first 4 rows $(h = 1, ..., 4)$, the second four rows $(h = 5, ..., 8)$, etc... In this way we have obtained contributions that are related to the intervals $D_{(1)} = (0 \dashv 0.2_.), D_{(2)} = (0.2 \dashv 0.4_.), ..., D_{(5)}(0.8 \dashv 1.0)$. Finally, Table 14 reports the decompositions of the Bonferroni index $V(Y) = 0.5225$ according to the five intervals $D_{(s)}$ related to the cumulative frequency $p_{h.}$, and to the five subpopulations. In each cell sl of Table 14, we have reported: the contributions $V_{(s)l.} = \sum_{h=1+4(s-1)}^{4s} \frac{1}{20} V_{hl.}$, and the corresponding within $V_{(s)lW}$ and between parts. Moreover, in each cell s of the marginal column we have reported the contribution $V_{(s)..} = \sum_{l=1}^{5} V_{(s)l.}$ and the corresponding within and between parts, and in cell l of the marginal row we have reported the contribution $V_{.l.} = \sum_{s=1}^{5} V_{(s)l.} = \sum_{h=1}^{r} (1/20)V_{hl.}$ and the corresponding within and between parts. Note that $V(Y) = \sum_{s=1}^{5} V_{(s)..} = \sum_{s=1}^{5} 0.2 \cdot \overline{V}_s$, where $\overline{V}_s = \frac{1}{4} \sum_{h=1+4(s-1)}^{4s} V_h, s = 1, ..., 5$. This means that we can represent $V(Y)$ as the sum of the areas of 5 rectangles with bases 0.2 and heights $\overline{V}_s$. This representation can be extended to the Gini and Zenga indexes, too.

TABLE 26. - *Calculation of* $[\frac{1}{20} V_{h4.}, V_{.4.}]$ *and of* $[\frac{1}{20} B_{h4.}, B_{.4.}]$

$N = 20; M = 10; V_{h4.} = \frac{M - \bar{M}_{h4}}{M} p(4 \mid h); B_{h4.} = \frac{\overset{+}{M}_{h.} - \bar{M}_{h4}}{\overset{+}{M}_{h.}} p(4 \mid h)$

	a	b	c	$d = \frac{1}{20} c \cdot a$	e	f	$g = \frac{1}{20} f \cdot a$
h	$p(4 \mid h)$	$\bar{M}_{h4}$	$\frac{M-b}{M}$	$\frac{1}{20} V_{h4.}$	$\overset{+}{M}_{h.}$	$\frac{e-b}{e}$	$\frac{1}{20} B_{h4.}$
1	0	...	...	0	...	...	0
⋮	0	...		0			0
12	0	...		0	...	...	0
13	1/13	12	-0.2	-0.00077	17.428	0.3114	0.00120
14	2/14	12.5	-0.25	-0.00179	18.166	0.3119	0.00223
15	3/15	13	-0.3	-0.003	19	0.31579	0.00316
16	4/16	13.5	-0.35	-0.00437	20	0.325	0.00406
17	4/17	13.5	-0.35	-0.00412	21.333	0.36719	0.00432
18	4/18	13.5	-0.35	-0.00389	23.5	0.42553	0.00473
19	4/19	13.5	-0.35	-0.00368	29	0.53448	0.00563
20	4/20	13.5	-0.35	-0.0035	29	0.53448	0.00534
Tot.				-0.02512 $= V_{.4.}$			0.0307 $= B_{.4.}$

In the second numerical example we have a population of $N = 7$ units partitioned into $k = 3$ overlapping subpopulations: the sizes of the three subpopulations are $n_{.1} = 2, n_{.2} = 3$ and $n_{.3} = 2$; the variate Y assumes the following $r = 6$ distinct values $(y_1 = 1, y_2 = 3, y_3 = 4, y_4 = 6, y_5 = 7, y_6 = 10)$, and $(n_{1.} = 2, n_{2.} = 1, n_{3.} = = 1, n_{4.} = 1, n_{5.} = 1, n_{6.} = 1)$. In Table 17 we have reported the 6×3 bivariate distribution of this population. Many authors (Silber, 1989; Dagum, 1997a; Ogwang, 2014, 2016) utilized this numerical example for the illustration of their proposal of the decomposition by subpopulations of the Gini index. Now, we remark that Gini (1914) showed that $G(Y)$ can be provided by the ratio of the concentration area and $1/2$. In the same fundamental paper Gini showed that $G(Y)$ is also given by the ratio $G(Y) = \frac{R}{2M} = \frac{1}{2M} \cdot \frac{S}{N^2}$, where $R = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mid y_{(i)} - y_{(j)} \mid = \frac{1}{N^2} \sum_{h=1}^{r} \sum_{t=1}^{r} \mid y_h - y_t \mid n_{h.} n_{t.}$ is the Gini mean difference with replacement and S is the numerator of R. In this last formula for the calculation of $G(Y)$ the connections between the Gini (1914) relative point inequality $\rho(p_{(i)}) = (p_{(i)} - q_{(i)})/p_{(i)}$ and the Lorenz curve are lost.

There are many ways for the calculation of S, the more popular are:

$$S = \sum_{i=1}^{N} \sum_{j=1}^{N} \mid y_{(i)} - y_{(j)} \mid = 2 \sum_{i=1}^{N} y_{(i)} (2i - N - 1) \tag{76}$$

and

$$S = \sum_{h=1}^{r} \sum_{t=1}^{r} \mid y_h - y_t \mid n_{h.} n_{t.} = 2 \sum_{h=1}^{r} y_h \cdot n_{h.} (2P_{h.} - N - n_{h.}). \tag{77}$$

Putting (77) in the Gini concentration ratio $G(Y) = \frac{R}{2M} = \frac{1}{2M} \cdot \frac{S}{N^2}$, gives:

$$G(Y) = \sum_{h=1}^{r} \left\{ \frac{y_h}{T} (2P_{h.} - N - n_{h.}) \right\} \frac{n_{h.}}{N}. \tag{78}$$

TABLE 27. - *Calculation of* $G(Y) = \sum_{h=1}^{6} \left\{ \frac{y_h}{32} (2P_{h.} - 7 - n_{h.}) \right\} \frac{n_{h.}}{7}$; $T = 32, N = 7$

	a		b	c	d	e		
h	$\frac{y_h}{32}$	$P_{h.}$	$2P_{h.} - 7 - n_{h.}$	$a \cdot b$	$\frac{n_{h.}}{7}$	$c \cdot d$	G_h	$G_h \cdot d$
1	$\frac{1}{32}$	2	-5	-0.15625	2/7	-0.04464	0.33482	0.09566
2	$\frac{3}{32}$	3	-2	-0.1875	1/7	-0.02678	0.54464	0.07781
3	$\frac{4}{32}$	4	0	0	1/7	0	0.58036	0.08291
4	$\frac{6}{32}$	5	2	$+0.375$	1/7	0.05357	0.49107	0.07015
5	$\frac{7}{32}$	6	4	$+0.875$	1/7	0.125	0.33928	0.04847
6	$\frac{10}{32}$	7	6	$+1.875$	1/7	0.26786	0	0
Tot.					1	0.375		0.375
						$= G(Y)$		$= G(Y)$

For the second numerical example, we have reported in Table 27 the calculation of $G(Y) = 0.375$ given by (78), and the values of $G(Y) = \sum_{h=1}^{6} G_h \cdot (n_{h.}/7) = 0.375$ already computed in Table 19. It is important to note that while,

$$\sum_{h=1}^{6} \left\{ \frac{y_h}{32} (2P_{h.} - 7 - n_{h.}) \right\} \frac{n_{h.}}{7} = \sum_{h=1}^{6} G_h \cdot (n_{h.}/7) = 0.375,$$

the differences between the values of the corresponding addends of the two sums are remarkable. In particular, for:

$$(h = 1, 2),\ \left\{ \frac{y_h}{32} (2P_{h.} - 7 - n_{h.}) \right\} \frac{n_{h.}}{7} < 0 < G_h \cdot \frac{n_{h.}}{7};$$

$$(h = 3),\ \left\{ \frac{y_h}{32} (2P_{h.} - 7 - n_{h.}) \right\} \frac{n_{h.}}{7} = 0 < G_h \cdot \frac{n_{h.}}{7}.$$

This means that the values given by $\left\{ \frac{y_h}{32} (2P_{h.} - 7 - n_{h.}) \right\} \frac{n_{h.}}{7}$ are not related to the ''corresponding'' areas of the Lorenz curve. Consequently, it is not possible to give, to these values, a very clear and intuitive interpretation.

For the first numerical example we have reported in Table 28, for each $s = 1, ..., 5$, the values $A_{(s)..} = \sum_{h=1+4(s-1)}^{4s} \left\{ \frac{y_h}{200} (2P_{h.} - 20 - 1) \right\} \frac{1}{20}$, and the values $C_{(s)..} = \sum_{h=1+4(s-1)}^{4s} G_h \frac{1}{20}$ already computed in Table 13. Table 28 confirms that, also in the case of the population of $N = 20$ units, the 5 values given by $A_{(s)..}$ are not related to the corresponding 5 values given by $C_{(s)..}$.

TABLE 28. - *Calculation of* $G(Y) = \sum_{s=1}^{5} A_{(s)..}$ *and of* $G(Y) = \sum_{s=1}^{5} C_{(s)}$

s	$D_{(s)}$	$A_{(s)..}$	$C_{(s)..}$
1	0.0 ⊣ 0.2	−0.0215	+0.045
2	0.2 ⊣ 0.4	−0.0415	+0.093
3	0.4 ⊣ 0.6	+0.0025	+0.109
4	0.6 ⊣ 0.8	+0.1105	+0.093
5	0.8 ⊣ 1.0	0.3300	+0.040
$G(Y)$		0.38	0.38

We remark that the contributions of some subpopulations to the point and synthetic Bonferroni and Gini measures are negative. Viceversa, in the case of the Zenga index the contributions of the subpopulations cannot assume negative values.

We conclude this paper pointing out that the aggregation of the units of the population in groups delimitated by the quintiles (quartiles, ...) has no effect on the values of the contributions $[V_{.l.}(Y),\ C_{.l.}(Y),\ B_{.l.}(Y)]$ of the subpopulations to the three synthetic measures.

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